

FUNCTOR CALCULUS AND THE DISCRIMINANT METHOD (SMOOTH MAPS TO THE PLANE AND PONTRYAGIN CLASSES, PART III)

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ABSTRACT. The discriminant method is a tool for describing the cohomology, or the homotopy type, of certain spaces of smooth maps with uncomplicated singularities from a smooth compact manifold L to \mathbb{R}^k . We recast some of it in the language of functor calculus. This reformulation allows us to use the discriminant method in a setting where we wish to impose conditions on the multilocal behavior of smooth maps $f: L \rightarrow \mathbb{R}^k$.

1. INTRODUCTION

Vassiliev's discriminant method is a tool for describing the cohomology, or the homotopy type, of certain spaces of smooth maps with uncomplicated singularities from a smooth compact manifold L to \mathbb{R}^k . Here we recast some of it in the language of functor calculus, more precisely, in the language of the manifold calculus which has been traditionally used to study spaces of smooth embeddings [28, 29, 10]. This reformulation allows us to use the discriminant method in situations where we wish to impose conditions on the multilocal behavior of smooth maps $f: L \rightarrow \mathbb{R}^k$, that is, the restrictions of f to small neighborhoods of finite sets in L .

We recall Vassiliev's "main theorem" [25, 26]. Fix $r \geq 0$ and let \mathfrak{X} be an open semi-algebraic subset of the finite dimensional vector space P of polynomial maps of degree $\leq r$ from \mathbb{R}^ℓ to \mathbb{R}^k . This is required to be invariant under the action (by pre-composition and truncation) of the Lie group G of invertible polynomial maps $\mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$ fixing the origin. Next, let L be a smooth manifold of dimension ℓ , without boundary for simplicity. A smooth map $f: L \rightarrow \mathbb{R}^k$ is considered to have an *inadmissible point*, or *\mathfrak{X} -inadmissible point*, at $x \in L$ if the r -th Taylor polynomial of f at x , in local coordinates centered at x , belongs to $P \setminus \mathfrak{X}$. The space $C^\infty(L, \mathbb{R}^k; \mathfrak{X})$ of smooth maps $L \rightarrow \mathbb{R}^k$ which are everywhere admissible comes with a jet prolongation map

$$(1.1) \quad C^\infty(L, \mathbb{R}^k; \mathfrak{X}) \longrightarrow \Gamma(L; \mathfrak{X}) .$$

Here $\Gamma(L; \mathfrak{X})$ is the section space of a fiber bundle on L with fibers homeomorphic to \mathfrak{X} ; the fiber at $x \in L$ is the space of the r -jets of \mathfrak{X} -admissible map germs $(L, x) \rightarrow \mathbb{R}^k$.

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Theorem 1.1. [25, 26] *If the codimension of $P \setminus \mathfrak{X}$ in P is $\geq \ell + 2$ everywhere, then (1.1) induces an isomorphism in integer cohomology. If the codimension is $\geq \ell + 3$ everywhere, then (1.1) is a homotopy equivalence.*

Vassiliev assumes that L is closed. That is the difficult case; the case where L is (connected and) noncompact follows from Gromov's general h -principle [11],[12]. Vassiliev also has a variant of the theorem as formulated above where L is compact with boundary, and the focus is on admissible smooth maps $L \rightarrow \mathbb{R}^k$ prescribed on/near ∂L . We review and remake Vassiliev's proof in this paper and we will also review and remake the case where L has a boundary.

In many applications r is positive, \mathfrak{X} contains all $f: \mathbb{R}^\ell \rightarrow \mathbb{R}^k$ in P which are regular at the origin (i.e., whose derivative at the origin is a *surjective* linear map $\mathbb{R}^\ell \rightarrow \mathbb{R}^k$), and furthermore \mathfrak{X} is invariant under composition with invertible polynomial maps in the *target* \mathbb{R}^k . In such cases it is justified to think of the theorem as a statement about spaces of maps with moderate singularities. But it is worth remembering that these extra assumptions are not conditions listed in the theorem. We have allowed $r = 0$. The case where $r = 0$ reduces to an almost trivial statement, with some unnecessary hypotheses, on the approximation of continuous maps by smooth maps; nevertheless Vassiliev's proof is interesting even in that case, as we will explain in a moment.

For closed L , let $\mathcal{O}(L)$ be the poset of open subsets of L , ordered by inclusion. An open set $U \in \mathcal{O}(L)$ is *tame* if it contains a compact codimension zero smooth submanifold K such that the inclusion of $K \setminus \partial K$ in K is isotopic to a diffeomorphism. Let $\mathcal{O}^t(L)$ be the full sub-poset consisting of the tame open sets. There are contravariant functors \mathcal{S} , respectively \mathcal{S}_Γ , from $\mathcal{O}^t(L)$ to chain complexes taking $U \in \mathcal{O}^t(L)$ to the singular chain complex of $C^\infty(U, \mathbb{R}^k; \mathfrak{X})$, respectively of $\Gamma(U; \mathfrak{X})$. We shall apply manifold calculus to \mathcal{S} and \mathcal{S}_Γ . In manifold calculus it is customary to consider contravariant functors (cofunctors) from $\mathcal{O}(L)$ or $\mathcal{O}^t(L)$ to the category of *spaces*. The values of \mathcal{S} and \mathcal{S}_Γ are chain complexes, which we can view as spaces by using the Kan-Dold equivalence between chain complexes graded over the positive integers and simplicial abelian groups. Alternatively, we can take the view that manifold calculus should be applicable to cofunctors from $\mathcal{O}(L)$ or $\mathcal{O}^t(L)$ to any model category.

In manifold calculus, a cofunctor \mathcal{F} from $\mathcal{O}^t(L)$ to a model category is considered *good* if it takes isotopy equivalences in $\mathcal{O}^t(L)$ to weak equivalences [29]. A good cofunctor on $\mathcal{O}^t(L)$ has best polynomial approximations $\mathcal{F} \rightarrow T_i \mathcal{F}$, also known as Taylor approximations. Without unraveling the deeper meaning of *polynomial cofunctor*, which has something to do with higher excision properties, we can define $T_i \mathcal{F}$ by

$$T_i \mathcal{F}(U) = \operatorname{holim}_{V \in \mathcal{O}^i(U)} \mathcal{F}(V)$$

where $\mathcal{O}^i(U)$ is the sub-poset of $\mathcal{O}^t(U)$ consisting of the open sets which are diffeomorphic to a disjoint union of at most i copies of \mathbb{R}^ℓ . In particular, $T_i \mathcal{F}$ is determined up to natural weak equivalence by the restriction $\mathcal{F}|_{\mathcal{O}^i(L)}$. We say informally that \mathcal{F} is *analytic* if the canonical map from $\mathcal{F}(U)$ to $\operatorname{holim}_i T_i \mathcal{F}(U)$ is a weak equivalence for every $U \in \mathcal{O}^t(L)$.

Theorem 1.2. *If the codimension of $P \setminus \mathfrak{X}$ in P is $\geq \ell + 2$ everywhere, then the cofunctors \mathcal{S} and \mathcal{S}_Γ on $\mathcal{O}^t(L)$ are analytic.*

We now explain how theorem 1.1 for closed L is a corollary of theorem 1.2. Jet prolongation induces a natural transformation $\mathcal{S} \rightarrow \mathcal{S}_\Gamma$. It is straightforward to verify that $\mathcal{S}(U) \rightarrow \mathcal{S}_\Gamma(U)$ is a chain equivalence if U belongs to $\mathcal{O}^i(L)$ for some i . Since both functors are analytic, this implies, by the homotopy invariance property of homotopy inverse limits, that $\mathcal{S}(U) \rightarrow \mathcal{S}_\Gamma(U)$ is a chain equivalence for all $U \in \mathcal{O}^t(L)$. In particular, this holds for $U = L$, which proves the first part of theorem 1.1. The second part of theorem 1.1 follows because source and target in (1.1) are simply-connected if the codimension of $P \setminus \mathfrak{X}$ in P is $\geq \ell + 3$ everywhere.

Our proof of theorem 1.2 uses the discriminant method and is not far removed from Vassiliev's argument for theorem 1.1. Vassiliev constructs two spectral sequences converging to the cohomology of source and target in theorem 1.1, respectively, and then proceeds by spectral sequence comparison. We shall see that his spectral sequences are the spectral sequences associated with the Taylor towers of $\mathcal{S}(L)$ and $\mathcal{S}_\Gamma(L)$.

Vassiliev's proof of theorem 1.1 suggests certain generalizations which are unfortunately concealed in the statement. In the formulation of theorem 1.2 these generalizations are more accessible, as we shall now explain.

On the category of finite nonempty sets and injective maps, we have a cofunctor $T \mapsto P^T$ where P is the vector space of polynomials mentioned above. Suppose given a subfunctor \mathfrak{X}_\bullet of $T \mapsto P^T$ such that \mathfrak{X}_T is open semi-algebraic in P^T , and invariant under the action of the Lie group G^T . Again, let L be a smooth manifold of dimension ℓ , without boundary for simplicity. A smooth map $f: L \rightarrow \mathbb{R}^k$ is considered to be \mathfrak{X}_\bullet -admissible if, for every finite nonempty subset T of L , the T -tuple of r -th Taylor polynomials of f at $t \in T$, in any local coordinates centered at t , belongs to \mathfrak{X}_T . Let $C^\infty(L, \mathbb{R}^k; \mathfrak{X}_\bullet)$ be the space of admissible smooth maps $L \rightarrow \mathbb{R}^k$ and let $\mathcal{S}(L)$ be its singular chain complex (recycled notation).

Theorem 1.3. *Suppose that the functor \mathfrak{X}_\bullet is coherently open and large enough (definitions 4.3 and 4.5). Then, for closed L , the cofunctor on $\mathcal{O}^t(L)$ defined by $U \mapsto \mathcal{S}(U)$ is analytic.*

By general manifold calculus principles, the theorem leads to a spectral sequence converging to the homology of $C^\infty(L, \mathbb{R}^k; \mathfrak{X}_\bullet)$, the spectral sequence associated with the Taylor tower of $\mathcal{S}(L)$. The p -th column of its E^1 -page is, broadly speaking, the homology of the space of decorated cardinality p subsets S of L . The decorations available for such an S are in a space homeomorphic to $\text{hocolim}_{T \subset S} \mathfrak{X}_T$. For details see remark 4.17.

Theorem 1.3 generalizes one half of theorem 1.2. Namely, for an open subset \mathfrak{X} of P as in theorem 1.2, we define \mathfrak{X}_\bullet by $\mathfrak{X}_T = \mathfrak{X}^T \subset P^T$. Then the functor \mathcal{S} of theorem 1.3 coincides with the functor \mathcal{S} of theorem 1.2.

Sections 2, 3 and 4 constitute the body of the paper. Some technical or historical issues have been relegated to a number of appendices. For the reader who is interested to see an application or a model corollary of theorem 1.3 which is not covered by theorem 1.2, we have written sections 5 and 6. Section 5 is a review of manifold calculus with an emphasis on symmetry, i.e., group actions on manifolds. Section 6 relies on section 5 and on some definitions and results from Part I [20].

This paper is the third part of a sequence which will probably have five parts. The first four of these five should be almost independent of each other. In particular

we think that this part III has something to offer by itself, although it is true that parts I and II [20, 21] provide additional motivation for it.

2. THE DISCRIMINANT METHOD AFTER VASSILIEV

Let L be a smooth compact manifold of dimension ℓ . We fix $k, r \geq 0$ and take P to be the vector space of polynomial maps of degree $\leq r$ from \mathbb{R}^ℓ to \mathbb{R}^k . As in the introduction, suppose that $\mathfrak{X} \subset P$ is an open semi-algebraic subset invariant under the action of the Lie group G .

Example 2.1. Popular choices for \mathfrak{X} that satisfy the codimension condition in theorem 1.1 are as follows. In one choice we take $k = 1$ and $r = 3$ so that P is the space of polynomial maps of degree at most 3 from \mathbb{R}^ℓ to \mathbb{R} . We let \mathfrak{X} consist of all $f \in P$ which at the origin are either nonsingular, or have a Morse singularity, or a birth-death singularity. This example was considered by Igusa [15, 16].

In another example we take $k = 2$ and $r = 4$. Let \mathfrak{X} consist of all $f \in P$ whose germ at the origin is in one of the six left-right equivalence classes *regular*, *fold*, *cuspidal*, *swallowtail*, *lips* and *beak-to-beak* as in [20].

Let $\varphi: L \rightarrow \mathbb{R}^k$ be a smooth map such that the r -jets of φ at all points $x \in \partial L$, in any local coordinates centered at x , belong to \mathfrak{X} . Let Sm be the affine space of all smooth maps $f: L \rightarrow \mathbb{R}^k$ which satisfy $j^r f|_{\partial L} = j^r \varphi|_{\partial L}$, where j^r is the r -jet prolongation for maps from L to \mathbb{R}^k . This condition implies that f and φ agree on ∂L . If $r > 0$, it is stronger since it prescribes certain (higher) partial derivatives of f involving directions normal to ∂L , at points $x \in \partial L$. We equip Sm with the Whitney C^∞ topology.

Let $\mathcal{W} := C^\infty(L, \mathbb{R}^k; \mathfrak{X}, \varphi) \subset Sm$ be the open subset consisting of all f which are \mathfrak{X} -admissible. Let $\Gamma(L; \mathfrak{X}, \varphi)$ be the space of sections of the jet bundle $J^r(L, \mathbb{R}^k) \rightarrow L$ which extend $j^r \varphi$ on ∂L and take values in the subbundle determined by \mathfrak{X} . We have the jet prolongation map

$$(2.1) \quad \mathcal{W} = C^\infty(L, \mathbb{R}^k; \mathfrak{X}, \varphi) \longrightarrow \Gamma(L; \mathfrak{X}, \varphi) .$$

The general (relative) form of theorem 1.1 is as follows.

Theorem 2.2. [25, 26] *If the codimension of $P \setminus \mathfrak{X}$ in P is $\geq \ell + 2$ everywhere, then (2.1) induces an isomorphism in integer cohomology. If the codimension is $\geq \ell + 3$ everywhere, then (2.1) is a homotopy equivalence.*

For the proof, we can assume that the double $L \cup_{\partial L} L$ is contained in a euclidean space \mathbb{R}^N as a nonsingular real algebraic subset of \mathbb{R}^N and $\partial L \subset L \cup_{\partial L} L$ is also a nonsingular real algebraic subset. This is justified by the Nash-Tognoli embedding theorem. See [3], especially [3, rmk 14.1.15]. We identify L with the first summand in $L \cup_{\partial L} L$. This makes L into a real semi-algebraic subset of \mathbb{R}^N . Indeed L is the closure of the union of some connected components of the semi-algebraic set $L \cup_{\partial L} L \setminus \partial L$. Any connected component of a semi-algebraic subset of \mathbb{R}^N is semi-algebraic [3, thm.2.4.5], and the closure of a semi-algebraic subset of \mathbb{R}^N is semi-algebraic [3, prop.2.2.2].

By appendix D, we may assume that $\varphi: L \rightarrow \mathbb{R}^k$ extends to a polynomial map $\mathbb{R}^N \rightarrow \mathbb{R}^k$. Also by appendix D, we may assume that r (in the description of P and $\mathfrak{X} \subset P$) is even and strictly positive.

Theorem 2.3. *There exists an ascending sequence $(A_i)_{i \in \mathbb{N}}$ of finite dimensional (dimension d_i) affine subspaces of Sm with the following properties.*

- (i) Dense: *the union of the A_i is dense in Sm ;*
- (ii) Algebraic: *every $f \in A_i$ extends to a polynomial map on \mathbb{R}^N ;*
- (iii) Tame: *for $f \in A_i$, the number of $x \in L$ where f is \mathfrak{X} -inadmissible is bounded above by a constant $\alpha_i \in \mathbb{N}$;*
- (iv) Interpolating: *for every i and every $T \subset L \setminus \partial L$ with $|T| \leq i$, the projection $A_i \rightarrow \prod_{x \in T} J_x^r(\mathbb{R}^\ell, \mathbb{R}^k) \cong P^T$ is onto.*

For a proof of this, see Appendix A. This follows Vassiliev in all essentials, but we have eliminated some complicated transversality arguments.

By the density property, the inclusion

$$\operatorname{colim}_i (A_i \cap \mathscr{W}) \longrightarrow \mathscr{W}$$

is a weak homotopy equivalence. Hence we can approximate the cohomology of \mathscr{W} with the cohomology of $A_i \cap \mathscr{W}$. The reduced cohomology $\tilde{H}^s(A_i \cap \mathscr{W})$ is isomorphic to the locally finite homology

$$H_{d_i-s-1}^{\text{lf}}(A_i \setminus \mathscr{W})$$

by a form of Poincaré-Alexander-Lefschetz duality in the finite dimensional affine space A_i . See appendix C. Here we are using the fact that

$$B_i = A_i \setminus \mathscr{W}$$

is an ENR, euclidean neighborhood retract, an essential condition for the duality statement. Indeed, B_i is a closed semialgebraic set in A_i . (Triangulation theorems in [3] for example imply that closed semialgebraic sets in euclidean spaces are ENRs.) Still following Vassiliev, we therefore focus on B_i and its locally finite homology. (This explains the expression *discriminant method*, since $B_i \subset A_i$ can be called the discriminant variety.)

We construct a “resolution”

$$(2.2) \quad RB_i \rightarrow B_i,$$

a proper homotopy equivalence of locally compact spaces, where RB_i admits a filtration which will help us to understand its homological properties. In detail, RB_i is the classifying space of a topological poset whose elements are pairs (f, T) where $f \in B_i$ and $T \subset L$ is a *bad event*, i.e., a finite nonempty set of inadmissible points for f . (We note that such a T satisfies $T \subset L \setminus \partial L$ since $B_i \subset A_i \subset Sm$.) The order relation is given by $(f, S) \leq (g, T)$ iff $f = g$ and $S \subset T$. There is an obvious metrizable topology on the underlying set where we say that a sequence $((f_j, S_j))_{j \in \mathbb{N}}$ converges to (f, S) if $(f_j)_{j \in \mathbb{N}}$ converges to f , in the Whitney topology on $C^\infty(L, \mathbb{R}^k)$, and $(S_j)_{j \in \mathbb{N}}$ converges to S in the Hausdorff topology. See appendix B. With this topology, the order relation “ \leq ” is closed. This leads to a canonical choice of topology on the classifying space of the topological poset (see appendix B again). The fiber of the projection

$$RB_i \longrightarrow B_i$$

at $f \in B_i$ is the classifying space of the finite poset of all subsets of $L \setminus \partial L$ which are bad events (hence finite and nonempty) for f . Since that poset has a maximal element, we learn from this that (2.2) has contractible and compact fibers. By appendix B, it is also a proper map of locally compact spaces.

Lemma 2.4. *The space RB_i is an ENR and the resolution map $RB_i \rightarrow B_i$ is a proper homotopy equivalence.*

Proof. It is shown in appendix B that RB_i is an ENR. The fibers of $RB_i \rightarrow B_i$ are simplices. This implies that $RB_i \rightarrow B_i$ is a *cell-like* map according to [17], and so by [17, Thm 1.2] it is a proper homotopy equivalence. \square

We filter RB_i as follows. The underlying topological poset is filtered such that (f, S) lives in the s -th stage of the poset, where $s = |S|$. This determines a filtration

$$F_1 RB_i \subset F_2 RB_i \subset F_3 RB_i \subset \cdots \subset RB_i$$

such that all points in the interior of a nondegenerate simplex determined by a diagram

$$(f, S_0) < (f, S_1) < \cdots < (f, S_{n-1}) < (f, S_n)$$

belong to $F_s RB_i$, where $s = |S_n|$.

Lemma 2.5. *Each $F_p RB_i$ is an ENR. The restricted resolution map $F_i RB_i \rightarrow B_i$ induces an isomorphism in locally finite homology in dimensions greater than $d_i - i$.*

Proof. It is shown in appendix B that $F_p RB_i$ is an ENR. Now fix $i > 0$. Let $C \subset B_i$ consist of all $f \in B_i$ which admit a bad event of cardinality $\geq i$. Let $T = \{1, 2, \dots, i\}$ and let

$$Z \subset A_i \times \text{emb}(T, L \setminus \partial L)$$

consist of all (f, e) such that $e(T)$ is a bad event for f . This has codimension $\geq i(\ell + 2)$ in $A_i \times \text{emb}(T, L \setminus \partial L)$ by property (iv) in theorem 2.3 (see A.6), and so has codimension $\geq 2i$ relative to A_i . The set Z is a semialgebraic set and it follows that its image in A_i , which is C , is also semialgebraic; see [3, Prop 2.2.7].

Now let $F_i RB_i|C$ be the portion of $F_i RB_i$ projecting to C . Then $F_i RB_i|C$ is the image of a map

$$Z \times \Delta^{i-1} \longrightarrow F_i RB_i$$

where Δ^{i-1} plays the role of classifying space of the poset $\{1, 2, 3, \dots, i\}$. (By appendix B, this map can also be interpreted as a semi-algebraic map.) The codimension of $Z \times \Delta^{i-1}$ relative to A_i is $\geq i + 1$. Therefore the codimension of $F_i RB_i|C$ relative to A_i is also $\geq i + 1$. As the projection $F_i RB_i|C \rightarrow C$ is onto by definition, it follows that the codimension of C in A_i is $\geq i + 1$.

Next, there is a commutative square

$$(2.3) \quad \begin{array}{ccc} F_i RB_i|C & \longrightarrow & F_i RB_i \\ \downarrow & & \downarrow \\ C & \longrightarrow & B_i \end{array}$$

By appendix B, it is a square of ENRs and proper maps. Let B'_i be the pushout of

$$C \longleftarrow F_i RB_i|C \longrightarrow F_i RB_i$$

so that there is a canonical map $B'_i \rightarrow B_i$. As $F_i RB_i|C \rightarrow F_i RB_i$ is a cofibration, it follows that B'_i is an ENR; see [14]. The fibers of $B'_i \rightarrow B_i$ over points of C are singletons, and the other fibers are standard simplices. This implies that $B'_i \rightarrow B_i$

is a cell-like map, and so by [17] it is a proper homotopy equivalence. Therefore it induces an isomorphism

$$(2.4) \quad H_*^{\text{ef}}(B'_i) \cong H_*^{\text{ef}}(B_i) .$$

From the construction of B'_i and the comparison with B_i , the square (2.3) determines a long exact Mayer-Vietoris sequence relating the groups

$$H_*^{\text{ef}}(B_i), \quad H_*^{\text{ef}}(F_i RB_i) \oplus H_*^{\text{ef}}(C), \quad H_*^{\text{ef}}(F_i RB_i | C).$$

With our (co-)dimension estimates this proves that $H_*^{\text{ef}}(F_i RB_i) \rightarrow H_*^{\text{ef}}(B_i)$ is an isomorphism when $* > d_i - i$. \square

Summing up the main insights so far, we have

$$\tilde{H}^s(A_i \cap \mathcal{W}) \cong H_{d_i-s-1}^{\text{ef}}(B_i) \cong H_{d_i-s-1}^{\text{ef}}(F_i RB_i)$$

if $s+1 < i$. Therefore we shall examine the locally finite homology of $F_i RB_i$ and use the filtration

$$F_1 RB_i \subset F_2 RB_i \subset F_3 RB_i \subset \cdots \subset F_i RB_i$$

for that. By appendix C, the locally finite homology of the pair $(F_p RB_i, F_{p-1} RB_i)$ is isomorphic to the locally finite homology of

$$F_p RB_i \setminus F_{p-1} RB_i$$

since $F_{p-1} RB_i$ is a closed sub-ENR of $F_p RB_i$. Elements of $F_p RB_i \setminus F_{p-1} RB_i$ have the form (f, S, x) where $f \in A_i$ and $S \subset L \setminus \partial L$ is a bad event for f with $|S| = p$, while x is an element of the simplex spanned by S . (The coordinate x is in the interior of the simplex.) There is a map

$$(2.5) \quad F_p RB_i \setminus F_{p-1} RB_i \longrightarrow \binom{L \setminus \partial L}{p}$$

whose target is the space of unordered configurations of p points in $L \setminus \partial L$. It is defined by $(f, S, x) \mapsto S$.

Theorem 2.6. *The map (2.5) is a locally trivial projection, for $0 < p \leq i$. The fibers are homeomorphic to $\mathbb{R}^b \times \mathbb{R}^{p-1} \times (P \setminus \mathfrak{X})^p$ where $b = d_i - p \dim(P)$.*

Proof. To show that it is locally trivial we factorize the map as follows:

$$(2.6) \quad F_p RB_i \setminus F_{p-1} RB_i \longrightarrow E_p \longrightarrow \binom{L \setminus \partial L}{p}.$$

Here E_p is the space of triples (h, S, x) where S is an unordered configuration of p points in $L \setminus \partial L$,

$$h \in \prod_{s \in S} J_s^r(L, \mathbb{R}^k)$$

has coordinates h_s which are all inadmissible (i.e., not in \mathfrak{X} when local coordinates centered at s are used) and x is an element in the interior of the simplex spanned by S . The left-hand arrow associates to (f, S, x) as above the triple (h, S, x) , where $h = j_S^r f$. The right-hand arrow is forgetful, and it is clearly a fiber bundle projection with fibers homeomorphic to $(P \setminus \mathfrak{X})^p \times \mathbb{R}^{p-1}$. It remains to show that the left-hand arrow in (2.6) is also locally trivial. Given $(h, S, x) \in E_p$, the portion of $F_p RB_i \setminus F_{p-1} RB_i$ being mapped to (h, S, x) is the affine subspace

$$(2.7) \quad \{f \in A_i \mid j_S^r f = h\} .$$

Most important is the observation that the affine space (2.7) is nonempty and has a dimension which is independent of (h, S, x) . Indeed, it is a translate of the kernel of the projection

$$A_i \longrightarrow \prod_{s \in S} J_s^r(L, \mathbb{R}^k) \cong P^S$$

which is assumed to be onto by the interpolation condition. The condition is applicable because $|S| = p \leq i$. \square

We now ask how the spectral sequence in locally finite homology determined by the filtrations

$$F_1 RB_i \subset F_2 RB_i \subset F_3 RB_i \subset \cdots \subset F_i RB_i$$

depends on i . (Note that theorem 2.6 describes the E^1 page of this spectral sequence and shows that the E^1 page is independent of i in the “active” region.) The affine quotient A_{i+1}/A_i is a vector space, assuming $A_i \neq \emptyset$. For $p \leq i$ there is a map

$$(2.8) \quad \Psi: F_p RB_{i+1} \longrightarrow A_{i+1}/A_i$$

defined by $(f, S, x) \mapsto [f]$ in the notation of the proof of theorem 2.6. We do not claim that Ψ is a bundle projection, but we still wish to have the homological corollaries.

Proposition 2.7. *The spectral sequences in locally finite homology determined by the filtrations*

$$F_1 RB_i \subset F_2 RB_i \subset F_3 RB_i \subset \cdots \subset F_i RB_i$$

and

$$F_1 RB_{i+1} \subset F_2 RB_{i+1} \subset F_3 RB_{i+1} \subset \cdots \subset F_i RB_{i+1}$$

are isomorphic up to a shift of $d_{i+1} - d_i$, equal to the dimension of A_{i+1}/A_i .

Proof. First we look at the fibers of (2.8). The fiber $\Psi^{-1}(0)$ is exactly $F_p RB_i$. The fiber $\Psi^{-1}(v)$ is like $F_p RB_i$ but constructed from the translate $A_i + \bar{v}$ of A_i in Sm , where $\bar{v} \in A_{i+1}$ represents v . Now $A_i + \bar{v}$ satisfies roughly the same conditions that A_i satisfies (e.g. the interpolation condition), so that we have a result for $\Psi^{-1}(v)$ identical to 2.6 for $\Psi^{-1}(0) = F_p RB_i$.

More generally, let us choose a triangulation of A_{i+1}/A_i by linear simplices. Then the above arguments show that, for simplices σ and τ in A_{i+1}/A_i with $\sigma \subset \tau$, the inclusion

$$\Psi^{-1}(\sigma) \longrightarrow \Psi^{-1}(\tau)$$

induces an isomorphism in locally finite homology. It follows immediately that there is a Leray-Serre type spectral sequence converging to the locally finite homology of the source $F_p RB_{i+1}$ of Ψ , with second page equal to the locally finite homology of A_{i+1}/A_i with coefficients in the locally finite homology of the fiber of Ψ . For obvious reasons this collapses and so we obtain

$$H_*^{\text{lf}}(F_p RB_{i+1}) \cong H_{*-d_{i+1}+d_i}^{\text{lf}}(F_p RB_i).$$

A similar relative argument shows

$$H_*^{\text{lf}}(F_p RB_{i+1}, F_s RB_{i+1}) \cong H_{*-d_{i+1}+d_i}^{\text{lf}}(F_p RB_i, F_s RB_i)$$

for $s < p$. Therefore by exact couple technology, the two spectral sequences in the lemma are isomorphic up to a shift as stated. \square

Remark 2.8. The above proposition 2.7 provides, among other things, a belated justification for the strategy which we adopted earlier in this section by claiming or assuming that $H^*(A_i \cap \mathcal{W})$ is a good approximation to $H^*(\mathcal{W})$. Now we can be more precise. There is a commutative ladder of homomorphisms

$$(2.9) \quad \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_{i+1} \cap \mathcal{W}) & \xleftarrow{H_{d_{i+1}-*-1}^{\text{eff}}} & (F_{i+1}RB_{i+1}) \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_i \cap \mathcal{W}) & \xleftarrow{H_{d_i-*-1}^{\text{eff}}} & (F_iRB_i) \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_{i-1} \cap \mathcal{W}) & \xleftarrow{H_{d_{i-1}-*-1}^{\text{eff}}} & (F_{i-1}RB_{i-1}) \\ \vdots & & \vdots \end{array}$$

The maps in the left-hand column are induced by the inclusions $A_i \rightarrow A_{i+1}$, and the maps in the right-hand column are as in proposition 2.7. The horizontal map in row i is the composition

$$H_{d_i-*-1}^{\text{eff}}(F_iRB_i) \longrightarrow H_{d_i-*-1}^{\text{eff}}(B_i) \xrightarrow{\cong} \tilde{H}^*(A_i \cap \mathcal{W})$$

By lemma 2.5, it is an isomorphism for $*+1 < i$. By the dimension formula in theorem 2.6, the arrow

$$H_{d_{i-1}-*-1}^{\text{eff}}(F_{i-1}RB_{i-1}) \longrightarrow H_{d_i-*-1}^{\text{eff}}(F_iRB_i)$$

in the right-hand column of the ladder is also an isomorphism when $*+1 < i$. (Any deviation must be reflected in the locally finite homology of $F_iRB_i \setminus F_{i-1}RB_i$. By theorem 2.6, this is the total space of a bundle and as such has dimension

$$\leq i\ell + (d_i - i \dim(P)) + i - 1 + i(\dim(P) - \ell - 2) = d_i - i - 1$$

where the summand $i\ell$ is the dimension of the base space of the bundle.) It follows that the arrow

$$\tilde{H}^*(A_i \cap \mathcal{W}) \longrightarrow \tilde{H}^*(A_{i-1} \cap \mathcal{W})$$

is an isomorphism for $* < i - 2$.

It follows also that the spectral sequence described in proposition 2.7 (stabilized with respect to i) converges to the reduced cohomology of \mathcal{W} . Setting this up as a homological spectral sequence in the fourth quadrant, we have

$$E_{p,q}^1 = H_{p+q+d_i-1}^{\text{eff}}(F_pRB_i \setminus F_{p-1}RB_i)$$

for some or any $i \geq p$. The differentials have the form

$$d^s : E_{p,q}^s \longrightarrow E_{p-s,q+s-1}^s .$$

The dimension of $F_pRB_i \setminus F_{p-1}RB_i$ is $\leq d_i - p - 1$, by theorem 2.6 again. Therefore all $E_{p,q}^1$ where $p + q + d_i - 1 > d_i - p - 1$ are zero. This means $E_{p,q}^1 = 0$ for $q > -2p$

which leads to the following picture of the E^1 page (fat dots for potentially nonzero positions):

$$\begin{array}{cccccccc}
 \circ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot
 \end{array}$$

By reflecting at the origin, marked by a circle \circ , we can also view this as a cohomology spectral sequence in the second quadrant.

This completes our (re)construction of Vassiliev's spectral sequence converging to $H^*(\mathcal{W})$. Vassiliev uses the same ideas to construct an analogous spectral sequence converging to the reduced cohomology of $\Gamma(L; \mathfrak{X}, \varphi)$. Jet prolongation induces a map between the two spectral sequences which specializes to an isomorphism of the E^1 pages. This implies that the jet prolongation map itself induces an isomorphism from $H^*(\Gamma(L; \mathfrak{X}, \varphi))$ to $H^*(\mathcal{W})$.

We shall not follow this path. Instead we will finish this section with a few observations on finiteness properties of $H^*(\mathcal{W})$ and the Vassiliev spectral sequence. Then in section 3 we shall use the spectral sequence directly to prove the first half of theorem 1.2. The proof of the second half uses an independent argument which is easier. Theorem 1.1 can then be deduced from theorem 1.2 as explained in the introduction.

Remark 2.9 (Finite generation). Let X be a closed semi-algebraic set in some \mathbb{R}^n . Then $H_*^{\text{lf}}(X)$ is finitely generated as a graded abelian group. To show this we identify the one-point compactification X^ω of X with the closure in \mathbb{R}^{n+1} of $f(X) \subset \mathbb{R}^{n+1}$, where $f: \mathbb{R}^n \rightarrow S^n \subset \mathbb{R}^{n+1}$ is the inverse of stereographic projection:

$$f(x) = sx + (0, 0, \dots, 0, 1 - s) \in S^n, \quad s = \frac{2}{1 + \|x\|^2}.$$

As f is an algebraic map, $f(X)$ is a semi-algebraic set and so is its closure in \mathbb{R}^{n+1} , by [3]. Therefore X^ω is a compact ENR and so we have

$$H_*^{\text{lf}}(X) \cong H_*^{\text{st}}(X^\omega, \omega) \cong H_*(X^\omega, \omega)$$

where H_*^{st} is Steenrod homology; see appendix C. The isomorphism between Steenrod homology and ordinary homology in this case comes from the fact that Steenrod homology satisfies the seven Eilenberg-Steenrod axioms, which characterize ordinary homology on compact ENRs.

The consequence for us is that B_i in the above description of Vassiliev's spectral sequence has finitely generated $H_*^{\text{lf}}(B_i)$. Also, RB_i and the subspaces $F_p RB_i$ were constructed as geometric realizations of simplicial semi-algebraic sets, with proper face and degeneracy operators, and satisfying an upper bound condition due to the tameness property in theorem 2.3. It follows that $H_*^{\text{lf}}(RB_i)$ and $H_*^{\text{lf}}(F_p RB_i)$ are finitely generated as graded abelian groups. Hence the spectral sequence described

in proposition 2.7 and remark 2.8 has an E^1 -page which is finitely generated in each bi-degree. Consequently $H^*(\mathcal{W})$ is finitely generated in each degree.

3. DISCRIMINANT METHOD AND MANIFOLD CALCULUS

Let $\mathcal{O}(L)$ be the poset of open subsets of L which contain ∂L , ordered by inclusion. An open set $U \in \mathcal{O}(L)$ is *tame* if it contains a compact codimension zero smooth submanifold K , containing ∂L , such that the inclusion of \check{K} in U is isotopic to a diffeomorphism (relative to ∂L), where $\check{K} = (K \setminus \partial K) \cup \partial L$. We write

$$\mathcal{O}^t(L) = \{U \in \mathcal{O}(L) \mid U \text{ is tame}\}.$$

Fix $r \geq 0$ and $\mathfrak{X} \subset P$ as in section 2. The definitions of $C^\infty(L, \mathbb{R}^k; \mathfrak{X}, \varphi)$ and $\Gamma(L; \mathfrak{X}, \varphi)$ can be extended in a straightforward manner to make contravariant functors on $\mathcal{O}(L)$ by

$$(3.1) \quad \mathcal{S}(U) = \text{singular chain complex of } C^\infty(U, \mathbb{R}^k; \mathfrak{X}, \varphi)$$

$$(3.2) \quad \mathcal{S}_\Gamma(U) = \text{singular chain complex of } \Gamma(U; \mathfrak{X}, \varphi).$$

The general form of theorem 1.2 is as follows.

Theorem 3.1. *If the codimension of \mathfrak{X} is $\geq \ell + 2$, then the cofunctors \mathcal{S} and \mathcal{S}_Γ on $\mathcal{O}^t(L)$ defined by (3.1) and (3.2) are analytic.*

The definition of *analytic* used here generalizes that given in the introduction. For $i \geq 0$ there is a subposet $\mathcal{O}i(L) \subset \mathcal{O}^t(L)$. It consists of the $U \in \mathcal{O}^t(L)$ which can be written as a disjoint union of an open collar on ∂L and an open subset of $L \setminus \partial L$ abstractly diffeomorphic to a disjoint union of $\leq i$ copies of \mathbb{R}^ℓ . By saying that a good cofunctor F on $\mathcal{O}^t(L)$ is analytic we mean that the canonical map

$$F(U) \longrightarrow \operatorname{holim}_i \operatorname{holim}_{U \supset V \in \mathcal{O}i} F(V)$$

is a weak homotopy equivalence.

For U in $\mathcal{O}^t(L)$, let $\kappa(U)$ be the poset, ordered by inclusion, of compact codimension zero smooth submanifolds of U containing ∂L . Let

$$\mathcal{W}(U) = C^\infty(U, \mathbb{R}^k; \mathfrak{X}, \varphi)$$

so that $\mathcal{W}(L) = \mathcal{W}$. Also for $K \in \kappa(U)$ let

$$\mathcal{W}_K = \{f \in \mathcal{S}m \mid f \text{ admissible on } K\}.$$

Lemma 3.2. *For $U \in \mathcal{O}^t(L)$, there is a chain of natural (in U) homotopy equivalences*

$$\mathcal{W}(U) \simeq \cdots \simeq \operatorname{holim}_{K \in \kappa(U)} \mathcal{W}_K.$$

There is also a chain of natural homotopy equivalences between the singular cochain complex of $\mathcal{W}(U)$ and the homotopy direct limit, over $K \in \kappa(U)$, of the singular cochain complexes of the \mathcal{W}_K .

Proof. For every $K \in \kappa(U)$, the restriction map $\mathcal{W}_K \rightarrow C^\infty(K, \mathbb{R}^k; \mathfrak{X}, \varphi)$ is a homotopy equivalence. (This uses a smooth form of the Tietze-Urysohn extension principle which goes back to Borel; see [4, 4.9].) Therefore it is enough to show that the map

$$(3.3) \quad C^\infty(U, \mathbb{R}^k; \mathfrak{X}, \varphi) \simeq \operatorname{holim}_{K \in \kappa(U)} C^\infty(K, \mathbb{R}^k; \mathfrak{X}, \varphi)$$

is a homotopy equivalence. Now let $K \in \kappa(U)$ be such that the inclusion $\check{K} \rightarrow U$ is isotopic in U to a diffeomorphism relative to ∂L . Let

$$(h_t : \check{K} \rightarrow U)_{t \in [0,1]}$$

be such an isotopy, so that $h_0 : \check{K} \rightarrow U$ is the inclusion. Then we have a map

$$C^\infty(K, \mathbb{R}^k; \mathfrak{X}, \varphi) \longrightarrow C^\infty(U, \mathbb{R}^k; \mathfrak{X}, \varphi)$$

given by composing with the map $U \rightarrow K$ obtained by inverting h_1 . This is easily seen to be a homotopy inverse for the restriction map

$$C^\infty(U, \mathbb{R}^k; \mathfrak{X}, \varphi) \longrightarrow C^\infty(K, \mathbb{R}^k; \mathfrak{X}, \varphi).$$

As the set $\kappa'(U)$ of those $K \in \kappa(U)$ for which the inclusion $\check{K} \rightarrow U$ is isotopic in U to a diffeomorphism (relative to ∂L) is cofinal in $\kappa(U)$, it follows that (3.3) is a homotopy equivalence.

The same reasoning shows that we have a chain of natural homotopy equivalences relating the singular chain complex of $C^\infty(U, \mathbb{R}^k; \mathfrak{X}, \varphi)$ to the homotopy inverse limit, over $K \in \kappa(U)$, of the singular chain complexes of \mathscr{W}_K . It does not matter much whether we take that homotopy inverse limit over $\kappa(U)$ or over $\kappa'(U)$, as $\kappa'(U)$ is cofinal in $\kappa(U)$. Since, for $K_1 \leq K_2 \in \kappa'(U)$, the restriction map

$$\mathscr{W}_{K_2} \rightarrow \mathscr{W}_{K_1}$$

is a homotopy equivalence, the induced map of singular chain or cochain complexes is also a homotopy equivalence. It follows that

$$\begin{aligned} & \operatorname{hom} \left(\operatorname{holim}_{K \in \kappa(U)} (\text{singular chain complex of } \mathscr{W}_K), \mathbb{Z} \right) \\ & \simeq \operatorname{hocolim}_{K \in \kappa(U)} (\text{singular cochain complex of } \mathscr{W}_K) . \end{aligned} \quad \square$$

In the following we apply the discriminant method to the spaces \mathscr{W}_K where possible, and draw conclusions about $\mathscr{W}(U)$ by means of lemma 3.2. We still have the affine spaces $A_i \subset Sm$ and the ENR subspaces $B_i = A_i \setminus \mathscr{W}$ from section 2. For $K \in \kappa(L)$, let

$$B_{i,K} = A_i \setminus \mathscr{W}_K .$$

Then $B_{i,K}$ is a closed subset of B_i . It consists of all $f \in A_i$ which have inadmissible points in K . There is probably no reason to think that it is always an ENR, but in the proof of the following lemma we identify many cases when it is.

Lemma 3.3. *There is a chain of natural homotopy equivalences relating the homotopy colimit, over $K \in \kappa(U)$, of the cochain complexes*

$$C_{d_i-* -1}^{\text{eff}}(B_{i,K})$$

to the homotopy colimit, over $K \in \kappa(U)$, of $\tilde{C}^(\mathscr{W}_K \cap A_i)$.*

Proof. To clarify notation, the degree 5 part of $C_{d_i-* -1}^{\text{eff}}(B_{i,K})$ is $C_{d_i-6}^{\text{eff}}(B_{i,K})$, for example. We speak of a *cochain* complex because differentials raise degree by one. It is not claimed that we have

$$C_{d_i-* -1}^{\text{eff}}(B_{i,K}) \simeq \tilde{C}^*(\mathscr{W}_K \cap A_i)$$

for every $K \in \kappa(U)$. Suppose however that K happens to be a semi-algebraic subset of L . Then $B_{i,K}$ is a semi-algebraic subset of A_i and therefore an ENR. By appendix C there is a chain of natural homotopy equivalences

$$C_{d_i-* -1}^{\text{eff}}(B_{i,K}) \longrightarrow \cdots \longleftarrow \tilde{C}^*(\mathcal{W}_K \cap A_i) .$$

It remains to note that the set of those K in $\kappa(U)$ which are semi-algebraic is cofinal in $\kappa(U)$. This is easy. For arbitrary $K \in \kappa(U)$, choose a smooth function $f: K \rightarrow \mathbb{R}$ having 0 as a regular value with preimage $\partial K \setminus \partial L$, and such that K is the preimage of $[0, \infty)$ under f . This f can be approximated by a polynomial function $g: K \rightarrow \mathbb{R}$. If the approximation is sharp enough (on values and first derivatives), then the preimage of $[0, \infty)$ under g is an element of $\kappa(U)$ close to K in the sense that a small isotopy inside U (rel ∂L) will move it to K . \square

Let us now fix K in $\kappa(L)$. Recall that there are a resolution $RB_i \rightarrow B_i$ and a filtration of RB_i by subspaces $F_p RB_i$. Let $RB_{i,K} \rightarrow B_{i,K}$ be defined similarly: $RB_{i,K}$ is the classifying space of a topological poset whose elements are pairs (f, S) where $f \in A_i$ and S is a bad event for f contained in K . Clearly $RB_{i,K}$ is a subspace of RB_i ; beware that it is not defined as the preimage of $B_{i,K}$ under the resolution map $RB_i \rightarrow B_i$. Also, we can construct a filtration $F_p RB_{i,K}$ of $RB_{i,K}$ by $F_p RB_{i,K} := F_p RB_i \cap RB_{i,K}$. There is a pullback square

$$(3.4) \quad \begin{array}{ccc} F_p RB_{i,K} \setminus F_{p-1} RB_{i,K} & \longrightarrow & F_p RB_i \setminus F_{p-1} RB_i \\ \downarrow & & \downarrow \\ \left(\begin{smallmatrix} K \setminus \partial L \\ p \end{smallmatrix} \right) & \longrightarrow & \left(\begin{smallmatrix} L \setminus \partial L \\ p \end{smallmatrix} \right) \end{array}$$

where the horizontal maps are inclusions and the right-hand vertical map is the one from theorem 2.6.

Proposition 3.4 (Omnibus). *Let $U \in \mathcal{O}^t(L)$. There is a cofinal subposet $\kappa^{\text{alg}}(U)$ of $\kappa(U)$ such that the following are satisfied for each $K \in \kappa^{\text{alg}}(U)$ and every $i > 0$:*

- (a) *The resolution map $RB_{i,K} \rightarrow B_{i,K}$ is a proper homotopy equivalence of ENRs.*
- (b) *Each $F_p RB_{i,K}$ is an ENR.*
- (c) *For $p \leq i$, the restricted resolution map $F_p RB_{i,K} \rightarrow B_{i,K}$ induces an isomorphism in locally finite homology in degrees greater than $d_i - p$.*
- (d) *The locally finite homology of $B_{i,K}$, $RB_{i,K}$ and $F_p RB_{i,K}$ is finitely generated as a graded abelian group.*

Proof. The cofinal subposet $\kappa^{\text{alg}}(U)$ is implicit in the proof of lemma 3.3. It consists of all $K \in \kappa(U)$ such that K is a semi-algebraic set in L . The proof of (a) is like the proof of lemma 2.4: $RB_{i,K}$ and $B_{i,K}$ are both ENRs, the fibers of $RB_{i,K} \rightarrow B_{i,K}$ are simplices, and so $RB_{i,K} \rightarrow B_{i,K}$ is a proper homotopy equivalence by [17]. The proofs of (b) and (c) are like the proof of lemma 2.5. The proof of (d) is as in remark 2.9. \square

Remark 3.5. Given $K \in \kappa^{alg}(U)$, there is a commutative ladder of homomorphisms

$$(3.5) \quad \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_{i+1} \cap \mathcal{W}_K) & \longleftarrow & H_{d_{i+1}-*-1}^{\text{ef}}(F_{i+1}RB_{i+1,K}) \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_i \cap \mathcal{W}_K) & \longleftarrow & H_{d_i-*-1}^{\text{ef}}(F_iRB_{i,K}) \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_{i-1} \cap \mathcal{W}_K) & \longleftarrow & H_{d_{i-1}-*-1}^{\text{ef}}(F_{i-1}RB_{i-1,K}) \\ \downarrow & & \uparrow \\ \vdots & & \vdots \end{array}$$

It is analogous to (2.9). By property (c) of the Omnibus proposition, the horizontal map in row i is an isomorphism for $*+1 < i$. By the pullback square (3.4), the arrow

$$H_{d_{i-1}-*-1}^{\text{ef}}(F_{i-1}RB_{i-1,K}) \longrightarrow H_{d_i-*-1}^{\text{ef}}(F_iRB_{i,K})$$

in the right-hand column of the ladder is also an isomorphism when $*+1 < i$. It follows that the arrow

$$H^*(A_i \cap \mathcal{W}_K) \longrightarrow H^*(A_{i-1} \cap \mathcal{W}_K)$$

is an isomorphism for $* < i-2$.

For an integer $p \geq 1$, let $E_{i,p}$ be the covariant functor from $\mathcal{O}^t(L)$ to cochain complexes taking $U \in \mathcal{O}^t(L)$ to

$$\text{colim}_{K \in \kappa(U)} C_{d_i-*-1}^{\text{ef}}(F_pRB_{i,K}) .$$

We also write $E_{i,\infty}$ for the (monotone) union or direct limit of the $E_{i,p}$, and $E_{i,0} = 0$ by convention. Since all cochain maps used in constructing the above colimit are cofibrations, i.e., degreewise split injective, the colimit can also be regarded as a homotopy colimit. There are cofibrations $E_{i,p} \subset E_{i,p+1}$ for $p \geq 0$.

Theorem 3.6. *For p with $1 \leq p \leq i$, the covariant functor $E_{i,p}/E_{i,p-1}$ is homogeneous of degree p .*

Proof. We know already that $E_{i,p}/E_{i,p-1}$ can be identified (up to natural chain equivalence) with the covariant functor on $\mathcal{O}^t(L)$ given by

$$(3.6) \quad U \mapsto \text{colim}_{K \in \kappa(U)} C_{d_i-*-1}^{\text{ef}}(F_pRB_{i,K} \setminus F_{p-1}RB_{i,K}) .$$

By (2.5) and theorem 2.6 and the pullback square (3.4), there is a bundle projection

$$(3.7) \quad F_pRB_{i,K} \setminus F_{p-1}RB_{i,K} \longrightarrow \binom{K \setminus \partial L}{p}$$

whose fibers are described in theorem 2.6. Taking colimits over $K \in \kappa(U)$, we still have a bundle projection

$$(3.8) \quad F_pRB_{i,U} \setminus F_{p-1}RB_{i,U} \longrightarrow \binom{U \setminus \partial L}{p}$$

where

$$F_p RB_{i,U} = \bigcup_{K \in \kappa(U)} F_p RB_{i,K} .$$

Therefore $E_{i,p}/E_{i,p-1}$ can be described as the functor taking U to the (re-indexed) complex of locally finite singular chains in the total space of (3.8) *supported over a compact set of the symmetric product U^p/Σ_p* . To show that this functor is homogeneous of degree p , we need to show that

- (i) it is polynomial of degree $\leq p$, according to a definition of *polynomial* for covariant functors which is given in appendix E;
- (ii) it takes every object in $\mathcal{O}m(L)$, with $m < p$, to a weakly contractible chain complex.

The “true reason” why these properties hold is that the functor (3.6), in the description just given, has the (dual of the) standard form of a homogeneous degree p functor given for example in [29]. But rather than trying to make a conversion, we give a direct argument.

Suppose therefore that U comes equipped with pairwise disjoint closed subsets C_0, \dots, C_p in $U \setminus \partial L$, as in appendix E. We assume that the C_t are pairwise disjoint tame co-handles. Let $S = \{0, 1, \dots, p\}$. For $T \subset S$ let $C_T = \bigcup_{t \in T} C_t$. In order to establish (i) we need to show that the S -cube of chain complexes

$$T \mapsto \frac{E_{i,p}}{E_{i,p-1}}(U \setminus C_T)$$

is cocartesian. This is equivalent to saying that the canonical map

$$\text{hocolim}_{\emptyset \neq T \subset S} \frac{E_{i,p}}{E_{i,p-1}}(U \setminus C_T) \longrightarrow \frac{E_{i,p}}{E_{i,p-1}}(U)$$

is a weak equivalence. Here we may adopt the interpretation suggested above, so that the target is the complex of locally finite singular chains in the total space of (3.8) supported over a compact set of U^p/Σ_p . Then each $(E_{i,p}/E_{i,p-1})(U \setminus C_T)$ can be identified with a chain subcomplex of the target, consisting of the locally finite singular chains supported over a compact set of $(U \setminus C_T)^p/\Sigma_p$. The homotopy colimit can be replaced by the internal sum of chain subcomplexes, because the system of these chain subcomplexes is Reedy cofibrant [13]. Therefore we have to show that the inclusion

$$(3.9) \quad \sum_{\emptyset \neq T \subset S} \frac{E_{i,p}}{E_{i,p-1}}(U \setminus C_T) \longrightarrow \frac{E_{i,p}}{E_{i,p-1}}(U)$$

is a weak homotopy equivalence. The target is still the complex of locally finite singular chains in the total space of (3.8) with support over a compact set of U^p/Σ_p . The source is the subcomplex of the target generated by all locally finite singular chains which, for some t , have support in a compact set of $(U \setminus C_t)^p/\Sigma_p$.

That (3.9) is a weak equivalence follows immediately from a subdivision argument [5, III.7] and the identity

$$\binom{U \setminus \partial L}{p} = \bigcup_{\emptyset \neq T \subset S} \binom{U \setminus (C_T \cup \partial L)}{p}.$$

This proves (i).

For property (ii), we suppose that $U \in \mathcal{O}m(L)$ for some $m < p$. This means that U is the disjoint union of an open collar on ∂L and at most m (without loss

of generality, exactly m) open subsets of L diffeomorphic to \mathbb{R}^ℓ . As we have to approximate U by compact subsets, we may focus on a compact K which is the disjoint union of a compact collar and m compact subsets of L diffeomorphic to D^ℓ . Then it suffices to show that the locally finite homology of

$$F_p RB_{i,K} \setminus F_{p-1} RB_{i,K}$$

is zero. For that it is enough, by C.5, to show that the locally compact space $F_p RB_{i,K} \setminus F_{p-1} RB_{i,K}$ can be written in the form $Y \times (0, 1]$ where Y is another locally compact space. For that it is enough, by local triviality of (3.7), to show that every connected component of

$$\binom{K \setminus \partial K}{p}$$

can be written in that form. We write $K \cong \partial L \times [0, 1] \cup D^\ell \times \{1, 2, \dots, m\}$. By selecting a connected component of the configuration space we are selecting a function

$$f: \{0, 1, 2, \dots, m\} \longrightarrow \mathbb{N} = \{0, 1, 2, \dots\}$$

which counts how many points of the configuration are in each component of K . In particular $f(0)$ counts the number of points from the configuration in the collar $\partial L \times [0, 1]$. Because $m < p$, we have either $f(0) > 0$ or $f(s) > 1$ for some s in $\{1, \dots, m\}$. These cases need to be treated separately. If $f(0) > 0$, we make a function

$$\binom{K \setminus \partial K}{p} \longrightarrow (0, 1]$$

by taking a configuration S to the maximum of all $t \in (0, 1]$ for which there is an element of S which belongs to the collar $\partial L \times [0, 1]$ and has second coordinate t . It is easy to show that the function is the projection of a product onto a factor. If $f(s) > 1$ for some $s \in \{1, \dots, m\}$, we make a function

$$\binom{K \setminus \partial K}{p} \longrightarrow (0, 1]$$

by taking a configuration S to the maximum of all $t \in (0, 1]$ for which there is an element of S which belongs to $D^\ell \times s$ and whose distance (in standard euclidean coordinates for the disk) from the center of the disk is t . Again it is easy to show that the function is the projection of a product onto a factor. \square

Proof of first part of 3.1. It suffices to show that $\mathcal{S}(U) \rightarrow T_p \mathcal{S}(U)$ is approximately p -connected for every $U \in \mathcal{O}^t(L)$. We fix $p > 0$ and some $i \gg p$.

We ought to look at $\mathcal{S}(U)$, the singular chain complex of $\mathcal{W}(U)$ as a contravariant functor of U . Because of lemmas 3.2 and 3.3 it is easier for us to work for a little while with the reduced singular cochain complex of $\mathcal{W}(U)$ as a *covariant* functor of U . We have the following diagram of natural transformations between covariant functors

$$\begin{array}{ccccc} \tilde{C}^*(\mathcal{W}(U)) & & & & E_{i,p}(U) \\ \uparrow \simeq & & & & \downarrow \\ \operatorname{hocolim}_K \tilde{C}^*(\mathcal{W}_K) & \longrightarrow & \operatorname{hocolim}_K \tilde{C}^*(\mathcal{W}_K \cap A_i) & \xrightarrow{\simeq} & E_{i,\infty}(U) \end{array}$$

where K runs through $\kappa(U)$. We abbreviate this to

$$Y \xleftarrow{\simeq} Y^\# \longrightarrow Y_i \xrightarrow{\simeq} E_{i,\infty} \longleftarrow E_{i,p}$$

suppressing the variable U . We apply the duality functor D (see appendix E) to obtain

$$DY \xrightarrow{\simeq} DY^\# \longleftarrow DY_i \xleftarrow{\simeq} DE_{i,\infty} \longrightarrow DE_{i,p}$$

and form the Taylor approximations T_p ,

$$\begin{array}{ccccccc} DY & \xrightarrow{\simeq} & DY^\# & \longleftarrow & DY_i & \xleftarrow{\simeq} & DE_{i,\infty} \longrightarrow DE_{i,p} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T_p DY & \xrightarrow{\simeq} & T_p DY^\# & \longleftarrow & T_p DY_i & \xleftarrow{\simeq} & T_p DE_{i,\infty} \longrightarrow T_p DE_{i,p} \end{array}$$

In the top row, all maps are approximately i -connected by remarks 3.5 and 2.8, except the one on the right which is approximately p -connected by the same remarks. In the bottom row, all maps except the one on the right are therefore approximately $(i - \ell p - p)$ -connected by lemma E.3 in appendix E. Noting that $T_p DE_{i,p} \simeq DE_{i,p} \simeq T_p DE_{i,i}$, we can view the one on the right as being induced by the approximately i -connected map $DE_{i,\infty} \rightarrow DE_{i,i}$. It is therefore also approximately $(i - \ell p - p)$ -connected by lemma E.3. It follows that the left-hand vertical arrow is approximately p -connected. Since DY is the reduced form of \mathcal{S} , this completes the proof. \square

It remains to prove the second part of theorem 3.1, which states that \mathcal{S}_Γ on $\mathcal{O}^t(L)$ is analytic. In appendix E, we have collected some definitions and facts related to the notions of analytic and polynomial functor. We proceed through a few examples where these will be used.

Example 3.7. Let Y be a space, homotopy equivalent to a CW-space. Let ψ be any map from ∂L to Y . We define F on $\mathcal{O}^t(L)$ by

$$F(U) = \text{map}(U, Y; \psi),$$

the space of maps $U \rightarrow Y$ which extend ψ . It is well known that F is polynomial of degree ≤ 1 , in the sense that

$$F(U) \longrightarrow \text{holim}_{W \in \mathcal{O}1(U)} F(W)$$

is a homotopy equivalence. Curiously this does not immediately tell us for which ρ and c the functor F is ρ -analytic with excess c (definition E.4). It is clear that, in the situation of definition E.4 and with our choice of functor F , the cube $F(U \setminus C_\bullet)$ is cartesian alias ∞ -cartesian if $j > 0$. So only the case $j = 0$ is of interest. In that case we write C for C_0 (the unique co-handle) and q for its codimension. The cube consists of a single map

$$F(U) \longrightarrow F(U \setminus C)$$

and we hope to be able to specify ρ , c in such a way that it is always $(c + \rho - q)$ -connected. Now suppose for simplicity that Y is ℓ -connected. Then we can show

that F is $(\ell+1)$ -analytic with excess 0. Indeed, there is a homotopy pullback square

$$\begin{array}{ccc} F(U) & \xrightarrow{\quad} & F(U \setminus C) \\ \downarrow & & \downarrow \\ \mathrm{map}(D^q, Y) & \xrightarrow{\quad} & \mathrm{map}(S^{q-1}, Y) \end{array}$$

because, up to homotopy equivalence, U is obtained from $U \setminus C$ by attaching a q -cell. The lower horizontal map is $(\ell+1-q)$ -connected, and so the upper horizontal map is also $(\ell+1-q)$ -connected. This is what we had to show.

This calculation is not useless. It follows from [30, 2.1] that if a good cofunctor F from $\mathcal{O}^t(L)$ to spaces is ρ -analytic with excess $c \geq 0$, then λF is also ρ -analytic with excess c , where $\lambda F(U)$ is the singular chain complex of $F(U)$. We emphasize that λF is typically not polynomial of degree ≤ 1 because the singular chain complex functor fails to preserve finite homotopy inverse limits (such as products and homotopy pullbacks). It is typically not polynomial of any degree. In the situation above where $F(U) = \mathrm{map}(U, Y; \psi)$ and Y is ℓ -connected, the Taylor tower of $\lambda F(U)$ converges to $\lambda F(U)$ for every $U \in \mathcal{O}^t(L)$, since every $U \in \mathcal{O}^t(L)$ admits a tame handle decomposition (relative to a collar on ∂L) with handles of index $\leq \ell$ only. For $U = L$, we obtain therefore a second quadrant spectral sequence converging to the “homotopy groups” of $\lambda F(L)$ which in this case we may interpret as the homology groups of $F(L)$. Some details on the E^1 -page of this spectral sequence can be found in remark 3.9.

Example 3.8. There is a variant of example 3.7 where we start with a fibration $E \rightarrow L$ and a section ψ of $E|_{\partial L}$. Let F be the cofunctor on $\mathcal{O}^t(L)$ defined by

$$F(U) = \Gamma(E|U; \psi)$$

(space of sections of $E|U$ which extend ψ on ∂L). If the fibers of $E \rightarrow L$ are ℓ -connected, then F is $(\ell+1)$ -analytic with excess 0 and so λF is also $(\ell+1)$ -analytic with excess 0. It follows that the Taylor tower of $\lambda F(U)$ converges to $\lambda F(U)$ for every $U \in \mathcal{O}^t(L)$. For $U = L$, this leads to a second quadrant spectral sequence converging to the homology groups of $F(L)$.

Proof of second part of theorem 3.1. The functor \mathcal{S}_Γ has the form λF described in example 3.8, for a fiber bundle $E \rightarrow L$ whose fibers are homeomorphic to \mathfrak{X} . The codimension of $P \setminus \mathfrak{X}$ in the vector space P is $\geq \ell+2$ everywhere, and so \mathfrak{X} is ℓ -connected. Hence $\mathcal{S}_\Gamma = \lambda F$ is $(\ell+1)$ -analytic with excess 0. The Taylor tower of $\mathcal{S}_\Gamma(U)$ therefore converges to $\mathcal{S}_\Gamma(U)$ for every $U \in \mathcal{O}^t(L)$. \square

Remark 3.9. The spectral sequence described in example 3.7, converging to the graded group $\pi_*(\lambda F(L)) \cong H_*(\mathrm{map}(L, Y; \psi))$, has the form

$$(3.10) \quad E_{-p,q}^1 = H_c^{-q}((\overset{L \setminus \partial L}{p}); \mathbf{U})$$

where H_c^* is generalized cohomology with compact supports and \mathbf{U} is a fibered spectrum. The fiber of \mathbf{U} over a configuration $S \subset L \setminus \partial L$, where $|S| = p$, is the spectrum

$$\mathbf{HZ} \wedge (Y^{*S} * S^0).$$

Here \mathbf{HZ} is the Eilenberg-MacLane spectrum and $Y^{*S} \cong Y * Y * \cdots * Y$ is the S -fold join power of Y . We use the base point of S^0 as a base point for the join $Y^{*S} * S^0$. The space $Y^{*S} * S^0$ can also be described as the total homotopy cofiber

of the S -cube of spaces $T \mapsto Y^T$, where $T \subset S$.

If Y is ℓ -connected, which we are assuming, then $Y^{*S} * S^0$ is $(\ell p + 2p - 1)$ -connected and so that cohomology group is trivial if $q \leq (\ell p + 2p - 1) - \ell p$, that is, if $q < 2p$. This leads to the following picture of the E^1 page (fat dots for potentially nonzero positions):

$$\begin{array}{cccccccc}
 \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet
 \end{array}$$

Formula (3.10) is a consequence of the description in [29] of the p -th homogeneous layer of a good cofunctor with values in based spaces or spectra, in our case the cofunctor λF . To use this we need to view λF as a cofunctor with spectrum values, using the stable Kan-Dold construction to transform chain complexes into spectra. See appendix E, proof of lemma E.3. Then, according to [29], the p -th homogeneous layer of λF , evaluated on L , is the spectrum of sections with compact support of a certain fibered spectrum \mathbf{V} on the configuration space

$$\binom{L \setminus \partial L}{p}.$$

The fiber of \mathbf{V} over a configuration S is the total homotopy fiber of the cube

$$T \mapsto \lambda F(N(T))$$

where $T \subset S$ and $N(T)$ is a tubular neighborhood of $T \cup \partial L$ in $L \setminus \partial L$. Here $\lambda F(N(T))$ should be regarded as a spectrum and this simplifies to $\mathbf{H}\mathbb{Z} \wedge (Y^T)_+$. In the category of spectra, total homotopy fibers of S -cubes agree with total homotopy cofibers up to a shift by $|S|$, so that $\mathbf{V} = \Omega^p \mathbf{U}$. Consequently π_{q-p} of the p -th homogeneous layer of $\lambda F(L)$ is identified with

$$H_c^{p-q}(\binom{L \setminus \partial L}{p}; \mathbf{V}) \cong H_c^{-q}(\binom{L \setminus \partial L}{p}; \mathbf{U}).$$

4. ADMISSIBLE MULTIJECTS

We now make the assumptions described just before theorem 1.3. In particular, for each finite nonempty set T we have an open semi-algebraic subset $\mathfrak{X}_T \subset P^T$, invariant under G^T . The assignment $T \mapsto \mathfrak{X}_T$ defines a subfunctor \mathfrak{X}_\bullet of $T \mapsto P^T$.

Definition 4.1. Let $f = (f_t)_{t \in T} \in P^T$. A nonempty subset $S \subset T$ is a *minimal bad event* for f if $(f_t)_{t \in S} \notin \mathfrak{X}_S$ and for every proper nonempty subset $R \subset S$ we have $(f_t)_{t \in R} \in \mathfrak{X}_R$. A nonempty subset $S \subset T$ is a *bad event* for $f = (f_t)_{t \in T}$ if it is a union of minimal bad events. The *complexity* of such a bad event S is the maximum of the integers j such that there exists a chain of bad subevents

$$S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_{j-1} \subsetneq S_j = S.$$

Evidently the complexity of a bad event S for $f = (f_t)_{t \in T}$ is not greater than $|S|$.

Example 4.2. A cheap way to construct \mathfrak{X}_\bullet as above is to specify an open semi-algebraic subset $\mathfrak{X} \subset P$, as in theorems 1.1 and 1.2, and to define

$$\mathfrak{X}_T := \mathfrak{X}^T \subset P^T .$$

In this situation, for $f \in Sm$, a subset T of $L \setminus \partial L$ is a bad event if and only if each point $t \in T$ is inadmissible for f . A bad event T is minimal if and only if $|T| = 1$. For another example, where the minimal bad events can have cardinality as large as 3, see [20, ch. 4].

Definition 4.3. The functor \mathfrak{X}_\bullet is *coherently open* if the following holds. Suppose given an open $U \subset \mathbb{R}^\ell$ and a finite nonempty $S \subset U$ and a smooth $f: U \rightarrow \mathbb{R}^k$ such that the multijet $j_S^r f$ belongs to \mathfrak{X}_S . Then there exist a neighborhood $V \subset U$ of S and a neighborhood W of f in $C^\infty(U, \mathbb{R}^k)$ such that for every $g \in W$ and finite nonempty subset T of V , the multijet $j_T^r g$ belongs to \mathfrak{X}_T .

Remark 4.4. Without loss of generality, V in definition 4.3 is a tubular neighborhood of S . Then every finite nonempty $T \subset V$ comes with a preferred map λ_T to S which takes $t \in T$ to the unique $s \in S$ in the same component of V . For those T where λ_T is injective, the coherence condition does not add anything to what we already have by assuming that \mathfrak{X}_\bullet is an open subfunctor of $T \mapsto P^T$. The interesting cases are those where λ_T is not injective.

The functor \mathfrak{X}_\bullet constructed in [20, ch. 4], where $\mathbb{R}^k = \mathbb{R}^2$, is coherently open. This is not explicitly verified in [20], but the proof is easy due to another property of \mathfrak{X}_\bullet which holds by construction: each $\mathfrak{X}_S \subset P^S$ is a union of finitely many left-right equivalence classes. Consequently, for prescribed $|S|$, only a finite list of germs $f: (U, S) \rightarrow \mathbb{R}^2$ with $j_S^r f \in \mathfrak{X}_S$ has to be tested for the condition in definition 4.3.

Definition 4.5. The functor \mathfrak{X}_\bullet is *large enough* if the following holds.

- For every finite nonempty set T , the elements of P^T having a bad event of cardinality p constitute a subset of codimension at least $p\ell + 2$ everywhere.
- There is an upper bound ζ for the cardinality of minimal bad events.
- There is a function $(p, j) \mapsto c(p, j) \in \mathbb{N}$, defined on pairs of integers (p, j) where $p \geq 1$ and $0 \leq j < p$, such that

$$\lim_{p \rightarrow \infty} c(p, j) - p\ell - j = \infty$$

and for every finite nonempty set T , the elements of P^T having a bad event of cardinality p and complexity j constitute a subset of codimension at least $c(p, j)$ everywhere.

Remark 4.6. Condition (i) in definition 4.5 ensures that the space of all smooth \mathfrak{X}_\bullet -admissible maps from L to \mathbb{R}^k (without any boundary conditions for the moment) is nonempty and even path connected. Namely, multijet transversality theorems imply that the set of $f \in C^\infty(L, \mathbb{R}^k)$ whose s -fold r -th order multijet prolongation (a section of a bundle on the configuration space of s -element subsets of L) makes a transverse intersection with the subbundle associated to

$$P^s \setminus \mathfrak{X}_{\{1, \dots, s\}} ,$$

for every $s \geq 0$, is dense in $C^\infty(L, \mathbb{R}^k)$. But a transverse intersection is an empty intersection by the codimension condition in (i); therefore such an f is admissible. More generally, the set of continuous paths $(f_t: L \rightarrow \mathbb{R}^k)_{t \in [0,1]}$ in $C^\infty(L, \mathbb{R}^k)$ whose s -fold r -th order multijet prolongation makes a transverse intersection with the

subbundle associated to $P^s \setminus \mathfrak{X}_{\{1, \dots, s\}}$, for every $s \geq 0$, is dense in the space of such paths. This remains true if we only allow paths (f_t) with fixed admissible f_0 and f_1 . Here again, a transverse intersection is an empty intersection by the codimension condition in (i). Therefore each f_t in a path satisfying the transversality condition is admissible.

Condition (ii) is there to ensure convergence of a spectral sequence generalizing Vassiliev's spectral sequence described in section 2. For any positive integer p , we define $\theta(p) = \min_{j < p} (c(p, j) - p\ell - j)$. Then the graph of

$$(4.1) \quad p \mapsto 2 - p - \theta(p)$$

is a vanishing curve in the E^1 -page of the spectral sequence. In the situation of section 2 alias example 4.2, where $\mathfrak{X}_T = \mathfrak{X}^T \subset P^T$ and \mathfrak{X} has codimension $\geq \ell + 2$ in P , we can take $c(p, j) = p(\ell + 2)$. Then the function (4.1) turns into $p \mapsto 1 - 2p$. Its graph is the vanishing line which we already saw in remark 2.8.

The following technical definition is of interest only when ∂L is nonempty.

Definition 4.7. Let T be a finite nonempty set, $(f_t)_{t \in T} \in P^T$ and $S \subset T$ a bad event for f . The coordinates f_t with $t \in S$ are elements of P . We say that they *participate* in the bad event S for $(f_t)_{t \in T}$.

An element g of P is considered *pure* or \mathfrak{X}_\bullet -*pure* if it has a neighborhood in P consisting of elements which do not participate in any bad event.

We now assume that a smooth map $\varphi: L \rightarrow \mathbb{R}^k$ has been selected which is \mathfrak{X}_\bullet -pure at every point in ∂L (that is, for every $x \in \partial L$, the jet $j_x^r \varphi$ is \mathfrak{X}_\bullet -pure, in local coordinates about x). This condition on φ is rather strong, for general \mathfrak{X}_\bullet and L with nonempty boundary.

Example 4.8. For \mathfrak{X}_\bullet as in the first part of example 4.2, a map $\varphi: L \rightarrow \mathbb{R}^k$ is \mathfrak{X}_\bullet -pure at all points of ∂L if and only if it is \mathfrak{X}_\bullet -admissible at all points of ∂L . For \mathfrak{X}_\bullet as in [20, ch. 4], a map $\varphi: L \rightarrow \mathbb{R}^k$ is \mathfrak{X}_\bullet -pure at all points of ∂L if and only if it is regular at all points of ∂L . This means that $D_x \varphi: T_x L \rightarrow \mathbb{R}^k$ is onto for every $x \in \partial L$. It does not mean that $D_x: T_x \partial L \rightarrow \mathbb{R}^k$ is onto for every $x \in \partial L$.

By analogy with section 2, let Sm be the affine space of all smooth maps f from L to \mathbb{R}^k which satisfy $j^r f|_{\partial L} = j^r \varphi|_{\partial L}$, where j^r is the r -jet prolongation for maps from L to \mathbb{R}^k . We equip Sm with the Whitney C^∞ topology.

Definition 4.9. For $f \in Sm$ and a nonempty finite subset T of L , we say that T is a (minimal) *bad event* for f if T is a (minimal) bad event for the multijet $j_T^r f$, expressed in local coordinates centered at the points of T . (This implies $T \subset L \setminus \partial L$.)

Let $\mathscr{W} := C^\infty(L, \mathbb{R}^k; \mathfrak{X}_\bullet, \varphi) \subset Sm$ be the open subset consisting of all f which are \mathfrak{X}_\bullet -admissible. Similarly, for every $U \in \mathcal{O}^t(L)$ we have $\mathscr{W}(U) := C^\infty(U, \mathbb{R}^k; \mathfrak{X}_\bullet, \varphi)$. Let $\mathcal{S}(U)$ be the singular chain complex of $\mathscr{W}(U)$. The general (relative) form of theorem 1.3 is as follows.

Theorem 4.10. *If \mathfrak{X}_\bullet is large enough and coherently open, then the functor \mathcal{S} is analytic.*

For the proof, we assume as in sections 2 and 3 that the double $L \cup_{\partial L} L$ is contained in a euclidean space \mathbb{R}^N as a nonsingular real algebraic subset of \mathbb{R}^N and $\partial L \subset L \cup_{\partial L} L$ is also a nonsingular real algebraic subset. We identify L with the

first summand in $L \cup_{\partial L} L$. We may assume that $\varphi: L \rightarrow \mathbb{R}^k$ extends to a polynomial map $\mathbb{R}^N \rightarrow \mathbb{R}^k$, and that r is even and strictly positive.

Theorem 4.11. *There exists an ascending sequence $(A_i)_{i \in \mathbb{N}}$ of finite dimensional (dimension d_i) affine subspaces of Sm with the following properties.*

- (i) Dense: *the union of the A_i is dense in Sm ;*
- (ii) Algebraic: *every $f \in A_i$ extends to a polynomial map on \mathbb{R}^N ;*
- (iii) Tame: *for $f \in A_i$, the cardinality of any bad event for f is bounded above by a constant $\alpha_i \in \mathbb{N}$;*
- (iv) Interpolating: *for every i and every $T \subset L \setminus \partial L$ with $|T| \leq i$, the projection $A_i \rightarrow \prod_{x \in T} J_x^r(\mathbb{R}^\ell, \mathbb{R}^k) \cong P^T$ is onto.*

As in section 2 we define $B_i = A_i \setminus \mathcal{W}$ and construct a “resolution”

$$(4.2) \quad RB_i \rightarrow B_i.$$

In detail, RB_i is the classifying space of a topological poset whose elements are pairs (f, T) where $f \in B_i$ and $T \subset L$ is a bad event for f . The order relation is given by $(f, S) \leq (g, T)$ iff $f = g$ and $S \subset T$.

Lemma 4.12. *The space RB_i is an ENR and the resolution map $RB_i \rightarrow B_i$ is a proper homotopy equivalence.*

Proof. It is shown in appendix B that RB_i is an ENR. Each fiber of $RB_i \rightarrow B_i$ is a classifying space of a finite poset with maximal element, hence a contractible and compact simplicial complex. This implies that $RB_i \rightarrow B_i$ is a *cell-like* map and therefore a proper homotopy equivalence. \square

We filter RB_i as follows. The underlying topological poset is filtered such that (f, S) lives in the $|S|$ -th stage of the poset. This determines a filtration

$$F_1 RB_i \subset F_2 RB_i \subset F_3 RB_i \subset \cdots \subset RB_i$$

such that all points in the interior of a nondegenerate simplex determined by a diagram

$$(f, S_0) < (f, S_1) < \cdots < (f, S_{n-1}) < (f, S_n)$$

belong to $F_{|S_n|} RB_i$.

Lemma 4.13. *Each $F_p RB_i$ is an ENR. The resolution map $F_i RB_i \rightarrow B_i$ induces an isomorphism in locally finite homology in dimensions $\geq d_i - \theta(i)$.*

Proof. It is shown in in appendix B that $F_p RB_i$ is an ENR. Fixing $i > 0$, let $C \subset B_i$ consist of all $f \in B_i$ which admit a bad event of cardinality $\geq i$. Let $T = \{1, 2, \dots, i\}$ and let

$$Z \subset A_i \times \text{emb}(T, L \setminus \partial L)$$

consist of all (f, e) such that $e(T)$ is a bad event for f . Write Z as a finite union of semi-algebraic subsets Z_Q where Q is a sub-poset of the set of nonempty subsets of $T \cong e(T)$, and $Z_Q \subset Z$ consists of the pairs (f, e) such that the poset of bad sub-events of $e(T)$ is exactly Q . Taking $j = \dim(BQ)$, we have that Z_Q has codimension $\geq c(i, j)$ in $A_i \times \text{emb}(T, L \setminus \partial L)$ by property (iv) in theorem 4.11 (see A.6), and so has codimension $\geq c(i, j) - i\ell$ relative to A_i . The set Z_Q is a semialgebraic set and it follows that its image in A_i is also semialgebraic.

Now let $F_i RB_i|C$ be the portion of $F_i RB_i$ projecting to C . Then $F_i RB_i|C$ is the image of a map

$$\bigcup_Q (Z_Q \times BQ) \longrightarrow F_i RB_i .$$

The codimension of $Z_Q \times BQ$ relative to A_i is $\geq c(i, j) - j - i\ell$ where $j = \dim(BQ)$. Hence the codimension of $F_i RB_i|C$ relative to A_i is $\geq \min_j (c(i, j) - i\ell - j) = \theta(i)$. As the projection $F_i RB_i|C \rightarrow C$ is onto, it follows that the codimension of C in A_i is $\geq \theta(i)$. As in the proof of lemma 2.5, it follows that $H_*^{\text{eff}}(F_i RB_i) \rightarrow H_*^{\text{eff}}(B_i)$ is an isomorphism when $*$ $> d_i - \theta(i)$. \square

There is a map

$$(4.3) \quad F_p RB_i \setminus F_{p-1} RB_i \longrightarrow \binom{L \setminus \partial L}{p}$$

whose target is the space of unordered configurations of p points in $L \setminus \partial L$.

Theorem 4.14. *The map (4.3) is a locally trivial projection, for $0 < p \leq i$. The fiber over a configuration S is homeomorphic to the space of triples (v, h, x) where*

- (a) $v \in \mathbb{R}^b$, with $b = d_i - p \dim(P)$;
- (b) $h \in \prod_{s \in S} J_s^r(L, \mathbb{R}^k)$ has S as a bad event;
- (c) x belongs to the open cone on the classifying space of the poset of proper subsets of S which are bad events for h .

Proof. To show that it is locally trivial we factorize the map as follows:

$$(4.4) \quad F_p RB_i \setminus F_{p-1} RB_i \longrightarrow E_p \longrightarrow \binom{L \setminus \partial L}{p}.$$

Here E_p is the space of triples (h, S, x) where S is an unordered configuration of p points in $L \setminus \partial L$,

$$h \in \prod_{s \in S} J_s^r(L, \mathbb{R}^k)$$

has S as a bad event and x is an element of the open cone mentioned in (c). The left-hand arrow associates to (f, S, x) as above the triple (h, S, x) , where $h = j_S^r f$. The right-hand arrow is forgetful, and it is clearly a fiber bundle projection with fiber over S homeomorphic to the space of pairs (h, x) satisfying the conditions (b) and (c). The left-hand arrow in (4.4) is an affine space bundle with fibers \mathbb{R}^b . \square

Proposition 4.15. *The spectral sequences in locally finite homology determined by the filtrations*

$$F_1 RB_i \subset F_2 RB_i \subset F_3 RB_i \subset \cdots \subset F_i RB_i$$

and

$$F_1 RB_{i+1} \subset F_2 RB_{i+1} \subset F_3 RB_{i+1} \subset \cdots \subset F_i RB_{i+1}$$

are isomorphic up to a shift of $d_{i+1} - d_i$, equal to the dimension of A_{i+1}/A_i .

Proof. Analogous to the proof of proposition 2.7. \square

Remark 4.16. There is a commutative ladder of homomorphisms

$$(4.5) \quad \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_{i+1} \cap \mathcal{W}) & \longleftarrow & H_{d_{i+1}-*-1}^{\text{ef}}(F_{i+1}RB_{i+1}) \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_i \cap \mathcal{W}) & \longleftarrow & H_{d_i-*-1}^{\text{ef}}(F_iRB_i) \\ \downarrow & & \uparrow \\ \tilde{H}^*(A_{i-1} \cap \mathcal{W}) & \longleftarrow & H_{d_{i-1}-*-1}^{\text{ef}}(F_{i-1}RB_{i-1}) \\ \vdots & & \vdots \end{array}$$

The maps in the left-hand column are induced by the inclusions $A_i \rightarrow A_{i+1}$, and the maps in the right-hand column are as in proposition 4.15. The horizontal map in row i is the composition

$$H_{d_i-*-1}^{\text{ef}}(F_iRB_i) \longrightarrow H_{d_i-*-1}^{\text{ef}}(B_i) \xrightarrow{\cong} \tilde{H}^*(A_i \cap \mathcal{W})$$

By lemma 4.13, it is an isomorphism for $*+1 < \theta(i)$. By the dimension formula in theorem 4.14, the arrow

$$H_{d_{i-1}-*-1}^{\text{ef}}(F_{i-1}RB_{i-1}) \longrightarrow H_{d_i-*-1}^{\text{ef}}(F_iRB_i)$$

is also an isomorphism when $*+1 < \theta(i)$. Indeed, any deviation would be accounted for by $F_iRB_i \setminus F_{i-1}RB_i$, which by theorem 4.14 has dimension

$$\leq i\ell + b + i \dim(P) - (\min_j c(i, j) - j) = i\ell + d_i - (\min_j c(i, j) - j) = d_i - \theta(i) .$$

It follows that the arrow $\tilde{H}^*(A_i \cap \mathcal{W}) \rightarrow \tilde{H}^*(A_{i-1} \cap \mathcal{W})$ is an isomorphism for $* < \theta(i) - 2$. Finally, one concludes that the spectral sequence described in proposition 4.15 (stabilized with respect to i) *converges* to the reduced cohomology of \mathcal{W} . The dimension of $F_pRB_i \setminus F_{p-1}RB_i$ is $\leq d_i - \theta(p)$, by theorem 4.14 again. Therefore all $E_{p,q}^1$ vanish where $p+q+d_i-1 > d_i - \theta(p)$, which means that $q \geq 2 - p - \theta(p)$.

We note that the analysis done of the spectral sequence converging to the reduced cohomology of $\mathcal{W}(U) = C^\infty(U, \mathbb{R}^k; \mathfrak{X}, \varphi)$ in section 3 generalizes painlessly to the more general setting of this section where $\mathcal{W}(U) = C^\infty(U, \mathbb{R}^k; \mathfrak{X}_\bullet, \varphi)$.

Proof of 4.10. It suffices to show that $\mathcal{S}(U) \rightarrow T_p\mathcal{S}(U)$ is approximately $\theta(p)$ -connected for every $U \in \mathcal{O}^t(L)$, where $\theta(p) = \min_j (c(p, j) - j - p\ell)$. We fix $p > 0$ and some $i \gg p$.

We ought to look at $\mathcal{S}(U)$, the singular chain complex of $\mathcal{W}(U)$ as a contravariant functor of U . Because of lemmas 3.2 and 3.3 it is easier for us to work for a little while with the reduced singular cochain complex of $\mathcal{W}(U)$ as a *covariant* functor of U . We have the following diagram of natural transformations between covariant

functors

$$\begin{array}{ccccc}
 \tilde{C}^*(\mathcal{W}(U)) & & & & E_{i,p}(U) \\
 \uparrow \simeq & & & & \downarrow \\
 \operatorname{hocolim}_K \tilde{C}^*(\mathcal{W}_K) & \longrightarrow & \operatorname{hocolim}_K \tilde{C}^*(\mathcal{W}_K \cap A_i) & \xrightarrow{\simeq} & E_{i,\infty}(U)
 \end{array}$$

where K runs through $\kappa(U)$. We abbreviate this to

$$Y \xleftarrow{\simeq} Y^\# \longrightarrow Y_i \xrightarrow{\simeq} E_{i,\infty} \longleftarrow E_{i,p}$$

suppressing the variable U . We apply the duality functor D (see appendix E) to obtain

$$DY \xrightarrow{\simeq} DY^\# \longleftarrow DY_i \xleftarrow{\simeq} DE_{i,\infty} \longrightarrow DE_{i,p}$$

and form the Taylor approximations T_p ,

$$\begin{array}{ccccccc}
 DY & \xrightarrow{\simeq} & DY^\# & \longleftarrow & DY_i & \xleftarrow{\simeq} & DE_{i,\infty} \longrightarrow DE_{i,p} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \simeq \\
 T_p DY & \xrightarrow{\simeq} & T_p DY^\# & \longleftarrow & T_p DY_i & \xleftarrow{\simeq} & T_p DE_{i,\infty} \longrightarrow T_p DE_{i,p} .
 \end{array}$$

In the top row, all maps are approximately $\theta(i)$ -connected by remarks 3.5 and 2.8, except the one on the right which is approximately $\theta(p)$ -connected by the same remarks. In the bottom row, all maps except the one on the right are therefore approximately $(\theta(i) - \ell p - p)$ -connected by lemma E.3 in appendix E. Noting that $T_p DE_{i,p} \simeq DE_{i,p} \simeq T_p DE_{i,i}$, we can view the one on the right as being induced by the approximately $\theta(i)$ -connected map $DE_{i,\infty} \rightarrow DE_{i,i}$. It is therefore also approximately $(\theta(i) - \ell p - p)$ -connected by lemma E.3. It follows that the left-hand vertical arrow is approximately $\theta(p)$ -connected. Since DY is the reduced form of \mathcal{S} , this completes the proof. \square

Remark 4.17. As mentioned in the introduction, the fact that \mathcal{S} is analytic leads via general manifold calculus principles to a spectral sequence converging to the homology of $\mathcal{S}(L)$. Of course we have already discussed that spectral sequence in 4.16, but we have not given the standard manifold calculus description. In order to use the description of the homogeneous layers in [29], we replace \mathcal{S} by the spectrum-valued functor F given by

$$F(U) = \mathbf{H}\mathbb{Z} \wedge \mathcal{W}(U)_+$$

for U in $\mathcal{O}^t(L)$. For simplicity we also assume that L is closed. The p -th homogeneous layer of F evaluated at L is the spectrum of sections with compact support of a fibered spectrum

$$(4.6) \quad \mathbf{\Lambda}(p) \longrightarrow \binom{L}{p} .$$

The fiber of $\mathbf{\Lambda}(p)$ over a configuration $S \subset L$, where $|S| = p$, is the total homotopy fiber of the S -cube

$$T \mapsto F(N(T))$$

where $N(T)$ is a tubular neighborhood of T . In the category of spectra, total homotopy fibers can be replaced by total homotopy cofibers up to a shift of p . As

$\mathcal{W}(N(\emptyset))$ is contractible and $\mathcal{W}(N(T)) \simeq J_T^r(L, \mathbb{R}^k; \mathfrak{X}_\bullet)$ for nonempty $T \subset S$, this means that the fiber of $\Lambda(p)$ over S can be described as

$$\Omega^p \mathbf{H}\mathbb{Z} \wedge \text{cone}\left(\text{hocolim}_{\emptyset \neq T \subset S} J_T^r(L, \mathbb{R}^k; \mathfrak{X}_\bullet) \rightarrow \star\right).$$

Here $J_T^r(L, \mathbb{R}^k; \mathfrak{X}_\bullet) \subset J_T^r(L, \mathbb{R}^k)$ consists of the admissible elements; it is homeomorphic to \mathfrak{X}_T . Finally using Poincaré duality, the spectrum of sections of (4.6) with compact support can be identified with a twisted smash product with base equal to that of (4.6) and fibers obtained from those in (4.6) by applying Ω^W , where W is one of the tangent spaces of the configuration space. Its homotopy group π_{q-p} is therefore the homology group (twisted by the orientation character of the configuration space) in degree $q - p + p + \ell p = q + \ell p$ of the projection map

$$Y \longrightarrow \binom{L}{p}$$

whose fiber over a configuration S is

$$Y_S = \text{hocolim}_{\emptyset \neq T \subset S} J_S^r(L, \mathbb{R}^k; \mathfrak{X}_\bullet).$$

(That homology group fits into a long exact sequence which also features the homology groups of Y and the configuration space.) This describes the term $E_{-p,q}^1$ of the spectral sequence. Clearly the emphasis here is on admissible multijets, whereas the description in remark 4.16 has the emphasis on bad multijets (bad events).

5. GROUP ACTIONS IN MANIFOLD CALCULUS

Following mainly [2], we revisit manifold calculus with an emphasis on symmetry. Both in sections 2, 3, 4 of this paper and in the foundational papers [29], [10], manifold calculus is concerned with contravariant functors from $\mathcal{O}(M)$ to Spaces. Here M is a *fixed* smooth manifold of dimension m and $\mathcal{O}(M)$ is the *discrete* poset of open subsets of M . The contravariant functors in question are required to take morphisms in $\mathcal{O}(M)$ which are *isotopy equivalences* (invertible up to isotopy in the larger category of smooth manifolds and codimension zero embeddings) to homotopy equivalences. Originally, the main examples of such functors were the functors $U \mapsto \text{emb}(U, N)$ for a fixed smooth N and $U \in \mathcal{O}(M)$, where $\text{emb}(U, N)$ is the space of smooth embeddings from U to N . In parallel, or possibly earlier, an operadic point of view specifically designed to deal with spaces of embeddings of 1-dimensional manifolds was developed by Kontsevich. It was noted by some, in particular by Arone and Turchin [1], that the operadic and manifold calculus viewpoints come together where we deal with contravariant functors from $\mathcal{O}(M)$ to spaces which factor with appropriate invariance properties through the category \mathcal{E} of smooth m -manifolds, without boundary to begin with, and smooth (codimension zero) embeddings.

One of the main points of [2] is that manifold calculus is worth re-developing for *continuous* contravariant functors from \mathcal{E} to Spaces. This makes the connection with operads quite clear because \mathcal{E} contains a copy of the operad of little m -balls. However, the operadic connection is not what we are after here. What we need here is the simple fact that, in manifold calculus according to [2], the Taylor approximations of a continuous contravariant functor F on \mathcal{E} are again continuous functors.

Example 5.1. (i) Let N be a fixed smooth manifold, $\dim(N) \geq m$. There is contravariant functor on \mathcal{E} defined by $L \mapsto \text{emb}(L, N)$.
 (ii) For L in \mathcal{E} let \mathcal{W}_L be the space of smooth maps $L \rightarrow \mathbb{R}^2$ satisfying the restrictions on singularities and multisingularities described in Part I [20]. Then $L \mapsto \mathcal{W}_L$ is again a contravariant functor.
 (iii) For L in \mathcal{E} let $J^r(L, \mathbb{R}^2) \rightarrow L$ be the jet bundle ($r \geq 4$ is enough). For a section σ of that jet bundle, we say that $x \in L$ is a singular point for σ if the linear part of the jet $\sigma(x)$ has rank < 2 . Let $\mathcal{W}_L^?$ be the space of continuous sections of that jet bundle which are integrable near and on the singular set, and which as such satisfy the same local and multilocal and boundary conditions as the elements of \mathcal{W}_L . So elements of $\mathcal{W}_L^?$ are a lot like elements of \mathcal{W}_L , but away from the singular set they provide not much more than an everywhere-rank-2 section of the bundle $\text{hom}(TL, \mathbb{R}^2)$, with no promise of integrability. Note that \mathcal{W}_L embeds in $\mathcal{W}_L^?$ by jet prolongation.

In these examples and in general, we work with continuous contravariant functors F on \mathcal{E} whose values are fibrant simplicial sets. We view the morphism spaces in \mathcal{E} as simplicial sets, too, and the continuity of F is expressed by saying that the evaluation

$$\text{mor}_{\mathcal{E}}(L, M) \times F(M) \longrightarrow F(L)$$

is a map of simplicial sets. (There are alternatives to the simplicial setup: we could assume or enforce that the values of F are compactly generated Hausdorff spaces, and arrange for \mathcal{E} and the target category of F to be enriched over compactly generated Hausdorff spaces. Our choice is not the bravest, but it allows us to say what is essential without having to make difficult decisions regarding topologies on mapping spaces.) In order to describe the Taylor approximations of F , we introduce certain small subcategories $\mathcal{E}k$ of \mathcal{E} , for $k = 0, 1, \dots$. Namely, $\mathcal{E}k$ is the full subcategory of \mathcal{E} spanned by the objects $\mathbb{R}^m \times \{1, 2, \dots, \ell\}$ for $0 \leq \ell \leq k$. (It has $k + 1$ objects.) Following [2], the k -th Taylor approximation of a continuous contravariant F can be (re)defined by

$$(5.1) \quad T_k F(L) := \text{holim}_{(U \rightarrow L) \text{ in } \mathcal{E}k \downarrow L} F(U) .$$

Here $\mathcal{E}k \downarrow L$ is the (comma) category whose objects are morphisms $U \rightarrow L$ in \mathcal{E} with fixed target L and variable source in $\mathcal{E}k$. The morphisms in $\mathcal{E}k \downarrow L$ are commutative triangles. Formula (5.1) wants to be the standard formula for the homotopy right Kan extension of $F|_{\mathcal{E}k}$ along the inclusion functor $\mathcal{E}k \rightarrow \mathcal{E}$. But a caveat is in order: \mathcal{E} and $\mathcal{E}k$ are enriched over simplicial sets and the homotopy inverse limit in the above definition uses these enrichments. Explicitly, $T_k F(L)$ is Tot of the cosimplicial space

$$(5.2) \quad [j] \mapsto \prod_{U_0, U_1, \dots, U_j} \text{map} \left(\left(\prod_{i=0}^{j-1} \text{mor}_{\mathcal{E}}(U_i, U_{i+1}) \right) \times \text{mor}_{\mathcal{E}}(U_j, L), F(U_0) \right)$$

where U_0, \dots, U_j are objects of $\mathcal{E}k$. The caveat is that $\text{map}(\dots, \dots)$ denotes the simplicial set of maps from one *simplicial* set to another (fibrant) simplicial set, not the simplicial set of maps from one discrete set to a simplicial set. It follows that $T_k F$ is again a *continuous* contravariant functor from \mathcal{E} to spaces.

Now suppose that L comes with an action of a topological group G by smooth maps, by which we mean a continuous group homomorphism from G to the group of

diffeomorphisms $L \rightarrow L$ (with the compact-open C^∞ topology). Then the singular simplicial set ΔG of G is a simplicial group which acts on L as an object of the category \mathcal{E} (enriched over simplicial sets). That action induces an action of ΔG on $T_k F(L)$, with the above definition (5.2).

We need to extend the above setup to include manifolds with boundary. Let K be a fixed smooth $(m-1)$ -manifold. We re-define \mathcal{E} as follows. The objects are smooth m -manifolds L with a specified diffeomorphism $K \rightarrow \partial L$. The morphisms from L_0 to L_1 are smooth embeddings $L_0 \rightarrow L_1$ making the triangle

$$\begin{array}{ccc} L_0 & \xrightarrow{\quad} & L_1 \\ & \nwarrow \quad \nearrow & \\ & K & \end{array}$$

commute. We redefine $\mathcal{E}k$ as the full subcategory of \mathcal{E} whose objects are the manifolds

$$\mathbb{R}^m \times \{1, \dots, \ell\} \amalg K \times [0, 1)$$

for $0 \leq \ell \leq k$, their boundaries being identified with K in the obvious way. For a continuous contravariant F from the redefined \mathcal{E} to spaces (simplicial sets), $T_k F$ can still be defined by formulae (5.1) and (5.1). Suppose that L in \mathcal{E} , as a smooth manifold, comes with an action of a topological group G fixing ∂L pointwise. Then the simplicial group ΔG acts on L as an object of \mathcal{E} . Therefore there is an induced action of ΔG on $T_k F(L)$.

We need more generality still in order to accommodate group actions which are nontrivial on boundaries. Suppose that the smooth manifold K above comes with an action of the topological group G by smooth maps. We create a category $\mathcal{E} \otimes G$, enriched over simplicial sets. It has the same objects as \mathcal{E} in our last definition. A 0-simplex in $\text{mor}_{\mathcal{E} \otimes G}(L_0, L_1)$ is a pair (f, z) where $z \in G$ and $f: L_0 \rightarrow L_1$ is a smooth embedding making the square

$$\begin{array}{ccc} L_0 & \xrightarrow{f} & L_1 \\ \uparrow & & \uparrow \\ K & \xrightarrow{\cdot z} & K \end{array}$$

commute. A p -simplex in $\text{mor}_{\mathcal{E} \otimes G}(L_0, L_1)$ is a pair (f, z) where z is a p -simplex in ΔG and f is a p -simplex in $\text{mor}_{\mathcal{E}}(L_0, L_1)$ such that the square

$$\begin{array}{ccc} \Delta^p \times L_0 & \xrightarrow{f} & L_1 \\ \uparrow & & \uparrow \\ \Delta^p \times K & \xrightarrow{\cdot z} & K \end{array}$$

commutes. Let $\mathcal{E}k \otimes G$ be the full subcategory of $\mathcal{E} \otimes G$ spanned by the objects $\mathbb{R}^m \times \{1, \dots, \ell\}$ where $0 \leq \ell \leq k$.

Let F be a continuous contravariant functor from $\mathcal{E} \otimes G$ to spaces. Clearly $\mathcal{E} \subset \mathcal{E} \otimes G$, so that we can view F as a contravariant functor on \mathcal{E} and apply functor calculus to it. Suppose that L is an object of \mathcal{E} which, as a smooth manifold, comes with an action of G by smooth maps which extends the prescribed action of G on $K \cong \partial L$. Then ΔG acts on L as an object of $\mathcal{E} \otimes G$, in other words, ΔG comes with a homomorphism to the (simplicial) automorphisms group of L in $\mathcal{E} \otimes G$. We would like to see a similar (analogous, compatible) action of ΔG on $T_k F(L)$. Such an

action is not clear from definition (5.1). Let us instead define $\lambda T_k F(L)$ as Tot of the cosimplicial space

$$(5.3) \quad [j] \mapsto \prod_{U_0, U_1, \dots, U_j} \text{map} \left(\left(\prod_{i=0}^{j-1} \text{mor}_{\mathcal{E} \otimes G}(U_i, U_{i+1}) \right) \times \text{mor}_{\mathcal{E} \otimes G}(U_j, L), F(U_0) \right)$$

where U_0, \dots, U_j are objects of $\mathcal{E}k \otimes G$. It is clear that ΔG acts on $\lambda T_k F(L)$. Therefore we require the following statement.

Proposition 5.2. *The restriction map $\lambda T_k F(L) \rightarrow T_k F(L)$ (induced by the inclusion $\mathcal{E} \rightarrow \mathcal{E} \otimes G$) is a homotopy equivalence.*

As a preparation for the proof, we introduce a rigidification of $\mathcal{E}k$. For objects U, V in $\mathcal{E}k$ let

$$\text{mor}_{\mathcal{E}}^r(U, V) \subset \text{mor}_{\mathcal{E}}(U, V)$$

consist of the morphisms f which on the copy of $K \times [0, 1)$ in U have the form $f(x, t) = (x, g(t))$ for some smooth embedding $g: [0, 1) \rightarrow [0, 1)$. (Strictly speaking this condition should be formulated simplicially.) We can make a subcategory of $\mathcal{E}k$ by allowing the same objects as before, and using the new morphism spaces $\text{mor}^r(U, V)$. Similarly, for U, V in $\mathcal{E}k \otimes G$ let

$$\text{mor}_{\mathcal{E}k \otimes G}^r(U, V) \subset \text{mor}_{\mathcal{E}k \otimes G}(U, V)$$

consist of the morphisms (f, z) which on the copy of $K \times [0, 1)$ in U have the form $f(x, t) = (z \cdot x, g(t))$ for some smooth embedding $g: [0, 1) \rightarrow [0, 1)$. Finally, fix a collar on the boundary of L once and for all. For U in $\mathcal{E}k$ let

$$\text{mor}_{\mathcal{E}}^r(U, L) \subset \text{mor}_{\mathcal{E}}(U, L)$$

consist of the morphisms f which take the copy of $K \times [0, 1)$ in U to the copy of $K \times [0, 1)$ alias collar in L and which, in the collar coordinates, have the form $f(x, t) = (x, g(t))$ for some smooth embedding $g: [0, 1) \rightarrow [0, 1)$. We assume that the action of G on L respects the collar in the sense that $z \cdot (x, t) = (z \cdot x, t)$ in collar coordinates, where $z \in G$ and $x \in K$, so that (x, t) denotes an element in L . Then let

$$\text{mor}_{\mathcal{E} \otimes G}^r(U, L) \subset \text{mor}_{\mathcal{E} \otimes G}(U, L)$$

consist of the morphisms (f, z) which take the copy of $K \times [0, 1)$ in U to the copy of $K \times [0, 1)$ alias collar in L and which, in the collar coordinates, have the form $f(x, t) = (z \cdot x, g(t))$ for some smooth embedding $g: [0, 1) \rightarrow [0, 1)$.

Lemma 5.3. *The rigidified morphism spaces have the following properties: (i) the inclusions*

$$\begin{aligned} \text{mor}_{\mathcal{E}}^r(U, V) &\rightarrow \text{mor}_{\mathcal{E}}(U, V), & \text{mor}_{\mathcal{E} \otimes G}^r(U, V) &\rightarrow \text{mor}_{\mathcal{E} \otimes G}(U, V) \\ \text{mor}_{\mathcal{E}}^r(U, L) &\rightarrow \text{mor}_{\mathcal{E}}(U, L), & \text{mor}_{\mathcal{E} \otimes G}^r(U, L) &\rightarrow \text{mor}_{\mathcal{E} \otimes G}(U, L) \end{aligned}$$

are homotopy equivalences; (ii) the projections

$$\begin{aligned} \text{mor}_{\mathcal{E} \otimes G}^r(U, V) &\longrightarrow \Delta G \times \text{mor}_{\mathcal{E}}^r(U, V), \\ \text{mor}_{\mathcal{E} \otimes G}^r(U, L) &\longrightarrow \Delta G \times \text{mor}_{\mathcal{E}}^r(U, L) \end{aligned}$$

are isomorphisms of simplicial sets; (iii) the action of ΔG on $\text{mor}_{\mathcal{E} \otimes G}(U, L)$ respects $\text{mor}_{\mathcal{E} \otimes G}^r(U, L)$.

Proof of proposition 5.2. We introduce the following abbreviations: $T_k^r F(L)$ for the rigidified version of $T_k F(L)$, where we use $\text{mor}_{\mathcal{E}}^r$ instead of $\text{mor}_{\mathcal{E}}$ throughout in the explicit formula (5.2), and $\lambda T_k^r F(L)$ for the rigidified version of $\lambda T_k F(L)$, where we use $\text{mor}_{\mathcal{E} \otimes G}^r$ instead of $\text{mor}_{\mathcal{E} \otimes G}$ throughout in the explicit formula (5.3). Then there is a commutative square of restriction maps

$$(5.4) \quad \begin{array}{ccc} \lambda T_k F(L) & \longrightarrow & T_k F(L) \\ \downarrow & & \downarrow \\ \lambda T_k^r F(L) & \longrightarrow & T_k^r F(L). \end{array}$$

The vertical arrows are homotopy equivalences. This uses a rule of thumb which says that if a map between cosimplicial spaces is a degreewise homotopy equivalence, then the induced map of Tots is a homotopy equivalence. Some conditions need to be satisfied: it is enough if in both cosimplicial spaces, all degeneracy operators are fibrations. (See [22, prop. A.1.(iv)] for the dual statement about simplicial spaces.) For the cosimplicial spaces that we are considering this is the case. Now it remains to show that the lower row in (5.4) is a homotopy equivalence. By lemma 5.3, we can write $\lambda T_k^r F(L)$ in the form

$$[j] \mapsto \prod_{U_0, U_1, \dots, U_j} \text{map} \left(\left(\prod_{i=0}^{j-1} \text{mor}_{\mathcal{E}}(U_i, U_{i+1}) \right) \times \text{mor}_{\mathcal{E}}(U_j, L) \times \Delta G^{j+1}, F(U_0) \right).$$

It is easy to recognize in this formula the homotopy limit of the following composition:

$$(\mathcal{E}^r k \downarrow L) \times \varepsilon \Delta G \xrightarrow{\text{proj}} \mathcal{E}^r k \downarrow L \xrightarrow{\text{forget}} \mathcal{E}^r k \xrightarrow{F|_{\mathcal{E}^r k}} \text{Simplicial Sets}.$$

Here $\mathcal{E}^r k$ is the rigidified version of $\mathcal{E} k$ (as in lemma 5.3) and $\varepsilon \Delta G$ is the simplicial category with simplicial object set ΔG and simplicial morphism set ΔG , acting on the simplicial object set by translation. By a general formula, this homotopy limit can be identified with the space of maps from $B(\varepsilon \Delta G)$ to the homotopy limit, which is $T_k^r F(L)$, of

$$\mathcal{E}^r k \downarrow L \xrightarrow{\text{forget}} \mathcal{E}^r k \xrightarrow{F|_{\mathcal{E}^r k}} \text{Simplicial Sets}.$$

Under this identification, the lower horizontal map in (5.4) is given by evaluating maps from $B(\varepsilon \Delta G)$ to $T_k^r F(L)$ at the base point. We finish by observing that $B(\varepsilon \Delta G)$ is contractible. \square

6. SMOOTH MAPS TO THE PLANE WITH MODERATE SINGULARITIES

Let \mathcal{W} be the space of smooth maps with moderate singularities (in source and target) from $D^n \times D^2$ to \mathbb{R}^2 described in Part I and example 5.1(ii). We shall often abbreviate $D_s = D^n \times D^2$ and $D_t = D^2$; the subscripts indicate *source* and *target*.

Theorem 6.1. *For n even and not too small, the restriction homomorphism*

$$H_{S^1}^{n-3}(\mathcal{W}, \star; \mathbb{Q}) \longrightarrow H_{S^1}^{n-3}(\mathcal{R}, \star; \mathbb{Q})$$

is zero; perhaps also with coefficients $\mathbb{Z}[1/2]$.

The theorem is intended to be a corollary of theorem 1.3. We recall the variant $\mathcal{W}^?$ of \mathcal{W} described in example 5.1(iii).

Lemma 6.2. *The forgetful map $\mathcal{W} \rightarrow \mathcal{W}^?$ induces a split surjection*

$$H_{S^1}^*(\mathcal{W}^?; \mathbb{Q}) \rightarrow H_{S^1}^*(\mathcal{W}; \mathbb{Q}).$$

Proof. Let \mathcal{E} be the category of smooth manifolds (with boundary and corners in the boundary) of dimension $n + 2$ whose boundary is identified with $K = \partial(D_s)$. As in section 5, the category \mathcal{E} is enriched over simplicial sets: $\text{mor}_{\mathcal{E}}(U, V)$ is the simplicial set of smooth embeddings extending the identity on boundaries. We extend \mathcal{W} and $\mathcal{W}^?$ to contravariant continuous functors F and $F^?$ on \mathcal{E} , with fibrant simplicial sets as values and with the simplicial interpretation of continuity. In previous sections our interest was in contravariant functors such as $U \mapsto S_*F(U)$ on $\mathcal{O}(M)$, where S_* is the singular chain complex. It is not a straightforward matter to recast these as functors on \mathcal{E} respecting suitable enrichments; but see [2]. Again we choose a more cowardly option, which is to recast S_* as a functor from simplicial sets to spectra, in such a way that the target category of spectra is enriched over simplicial sets. We will of course ensure that the values of the reformed S_* are generalized Eilenberg-MacLane spectra.

The spectra that we will allow are sequences $(X(i))_{i \geq 0}$ of based fibrant simplicial sets together with injective pointed maps

$$(\Delta^1/\partial\Delta^1) \wedge X(i) \longrightarrow X(i+1)$$

such that the adjoint $X(i) \rightarrow \text{map}_*(\Delta^1/\partial\Delta^1, X(i+1))$ is a homotopy equivalence, for every i . For spectra $\mathbf{X} = (X(i))_{i \geq 0}$ and $\mathbf{Y} = (Y(i))_{i \geq 0}$ of this type, the simplicial set $\text{mor}(\mathbf{X}, \mathbf{Y})$ has for its j -simplices sequences

$$(f_i: \Delta_+^j \wedge X(i) \longrightarrow Y(i))$$

of simplicial maps satisfying the evident compatibility condition. This makes our category \mathcal{SSP} of spectra into a category enriched over simplicial sets. Now we still wish to say how S_* can be viewed as an enriched functor from simplicial sets to \mathcal{SSP} . Let X be a simplicial set; let S_*X be the sequence of based fibrant simplicial sets whose i -th term is the free simplicial abelian group generated by

$$(\Delta^1/\partial\Delta^1)^{\wedge i} \wedge X_+$$

modulo the simplicial subgroup generated by the base point. The structure maps are clear. It is well known that the homotopy groups π_j of the i -th term of S_*X as defined are the homology groups $H_{j-i}(X; \mathbb{Z})$.

The main definitions of manifold calculus as outlined in section 5, such as Taylor approximations, remain meaningful for continuous functors from \mathcal{E} to \mathcal{SSP} . In particular, we can make something out of formula (5.2) in the case where the functor F takes values in \mathcal{SSP} . We only have to say what is meant by $\text{map}(X, \mathbf{Y})$ when X is a simplicial set and \mathbf{Y} is an object of \mathcal{SSP} . We mean by it the spectrum whose i -th term is the based simplicial set $\text{map}(X, Y(i))$. The structure maps are obvious. Statements such as *Taylor tower converges to functor* also remain meaningful and, as we saw in previous sections, can be formulated and sometimes proved without explicit recourse to continuity. (Typically, in the older setup where manifold calculus is about contravariant functors from the poset of open subsets of some M to some model category, there is a hidden continuity assumption in the requirement that F take isotopy equivalences to weak equivalences in the target category of F .)

Let $T_\infty S_*F$ be the top of the Taylor tower of F_* ; it can be defined as the homotopy

right Kan extension, along the inclusion $\bigcup_k \mathcal{E}k \rightarrow \mathcal{E}$, of the restriction $S_*F|_{\bigcup_k \mathcal{E}k}$. We have a commutative diagram

$$\begin{array}{ccc} S_*F(U) & \longrightarrow & S_*F^?(U) \\ \downarrow & & \downarrow \\ T_\infty S_*F(U) & \longrightarrow & T_\infty S_*F^?(U) \end{array}$$

By theorem 1.3 (version with boundary) and verification of the conditions for that theorem in [20], the left-hand column induces an isomorphism in homology. On the other hand, the lower row also induces an isomorphism in homology because $F(U) \rightarrow F^?(U)$ is a homotopy equivalence for every object U in $\bigcup_k \mathcal{E}k$, by inspection. Therefore the square provides a natural splitting up to homotopy for the inclusion-induced map $S_*F \rightarrow S_*F^?$. Because of the naturality, this is also compatible with the actions of ΔS^1 , the simplicial group associated with the Lie group S^1 . Here we do of course rely on the results of section 5. \square

Using lemma 6.2 we can prove theorem 6.1 by showing that the restriction homomorphism

$$H_{S^1}^{n-3}(\mathcal{W}^?, \star; \mathbb{Q}) \longrightarrow H_{S^1}^{n-3}(\mathcal{R}, \star; \mathbb{Q})$$

is zero. For that it is enough to show that the restriction

$$H_{S^1}^{n-3}(\mathcal{W}^?, \star; \mathbb{Q}) \longrightarrow H_{S^1}^{n-3}(\mathcal{V}, \star; \mathbb{Q})$$

is zero. (The inclusion $\mathcal{R} \rightarrow \mathcal{W}^?$ factors through \mathcal{V} because $\mathcal{W}^?$ is what it is.) Now \mathcal{V} is rationally a based sphere S^{n-3} alias Eilenberg-MacLane space, and so we know that the forgetful map

$$H_{S^1}^{n-3}(\mathcal{V}, \star; \mathbb{Q}) \longrightarrow H^{n-3}(\mathcal{V}; \mathbb{Q}) \cong \mathbb{Q}$$

is an isomorphism. Therefore it is enough to show that

$$H^{n-3}(\mathcal{W}^?; \mathbb{Q}) \longrightarrow H^{n-3}(\mathcal{V}; \mathbb{Q})$$

is zero. For that it is enough to show that the inclusion $\mathcal{V} \rightarrow \mathcal{W}^?$ is rationally nullhomotopic, which amounts to showing that a certain element of $\pi_{n-3}(\mathcal{W}^?)$ coming from $\pi_{n-3}(\mathcal{V})$ is zero or is a torsion element. At this point we need a good explicit description of that element.

The tangent bundle of S^n is classified by a map $S^n \rightarrow BO(n)$ or equivalently, by a map $g: S^{n-1} \rightarrow O(n)$ which we can assume to be smooth. We write S_δ^{n-1} for the standard sphere with fixed radius $\delta < 1$ and center 0 in the factor

$$0 \times \mathbb{R}^n \times 0 \subset \mathbb{R}^{n-3} \times (\mathbb{R}^n \times \mathbb{R}^2).$$

The normal bundle of S_δ^{n-1} is the sum of its 1-dimensional normal bundle in \mathbb{R}^n and the normal bundle of \mathbb{R}^n in $\mathbb{R}^{n-3} \times (\mathbb{R}^n \times \mathbb{R}^2)$, and so is trivialized in a preferred way. Change the trivialization using g and apply Pontryagin's construction to make a (smooth) map f from $\mathbb{R}^{n-3} \times (\mathbb{R}^n \times \mathbb{R}^2)$ to S^{n+1} , which is transverse to the antipode of the base point with transverse image S_δ^{n-1} , and so that the two normal framings of S_δ^{n-1} , determined respectively by g and the transversality property, agree. This f should take the complement (and the boundary) of $D^{n-3} \times D_\delta$ to the base point. The map f then represents an element of $\pi_{n-3}(\Omega^{n+2}S^n) \cong \pi_{2n-1}(S^n)$ which has Hopf invariant 2. Therefore $[f]$ generates the 1-dimensional vector space

$\pi_{2n-1}(S^n) \otimes \mathbb{Q}$. Now we have an embedding $\Omega_0^{n+2} S^n \rightarrow \mathcal{V}$ induced by an embedding $S^n \rightarrow V$ which takes $v \in S^n$ to the linear map

$$(z_1, \dots, z_n, x, y) \mapsto (v \cdot (z_1, \dots, z_n, x), y) .$$

Therefore we can and we will write $f: D^{n-3} \rightarrow \mathcal{V}$. Since f takes the entire boundary of D^{n-3} to the base point, it represents an element of $\pi_{n-3}(\mathcal{V})$. Since the embedding $\Omega_0^{n+2} S^n \rightarrow \mathcal{V}$ is a rational homotopy equivalence, we have shown:

Lemma 6.3. *The homotopy class (rel boundary) of $f: D^{n-3} \rightarrow \mathcal{V}$ generates the 1-dimensional vector space $\pi_{n-3}(\mathcal{V}) \otimes \mathbb{Q}$.*

Proof of theorem 6.1. We begin by producing a nullhomotopy rel boundary for the composition

$$D^{n-3} \xrightarrow{f} \mathcal{V} \longrightarrow \Omega^{n+2}(\text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2) \setminus 0) .$$

Equivalently, we produce an extension of $f: D^{n-3} \rightarrow \mathcal{V}$ to a map

$$\Phi: D^{n-3} \times [0, 1] \longrightarrow \Omega^{n+2}(\text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2) \setminus 0)$$

which takes the union of $\partial D^{n-3} \times [0, 1]$ and $D^{n-3} \times 1$ to the base point. The formula is $\Phi(p, t) := (1-t)f(p) + tv$ where v is the base point of $\text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2) \setminus 0$. The formula uses the vector space structure of $\Omega^{n+2}(\text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2))$. We like to write the adjoint of Φ as a map

$$\Phi^{\text{ad}}: D^{n-3} \times D_s \times [0, 1] \longrightarrow \text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2) \setminus 0 .$$

Later we will search for a lift of Φ , up to a homotopy rel boundary of $D^{n-3} \times [0, 1]$, to a map with target $\mathcal{W}^?$. For this we rely mainly on a good description of Φ^{ad} on a neighborhood U of the preimage of the rank 1 stratum in $\text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2)$. The map Φ^{ad} is transverse to the rank 1 stratum and the preimage of the rank 1 stratum is $S_\delta^{n-1} \times 1/2$. We can take U to be a tubular neighborhood of that. Next we determine the restriction of Φ^{ad} to $S_\delta^{n-1} \times 1/2$, as a continuous map

$$(6.1) \quad S_\delta^{n-1} \times 1/2 \longrightarrow \text{rank 1 stratum of } \text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2) .$$

That map is clearly constant, with value $(z_1, \dots, z_n, x, y) \mapsto (0, y)$. To understand Φ^{ad} sufficiently well on U , we also need to know the isomorphism φ , determined by the derivative of Φ^{ad} , from the normal bundle of $S_\delta^{n-1} \times 1/2$ in $D^{n-3} \times D_s \times [0, 1]$ to the pullback under (6.1) of the normal bundle of the rank 1 stratum in $\text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2)$. Since (6.1) is constant, the isomorphism φ amounts to a trivialization of the normal bundle of $S_\delta^{n-1} \times 1/2$ in $D^{n-3} \times D_s \times [0, 1]$. That trivialization is easy to recognize: it is the product of the trivialization of the normal bundle of S_δ^{n-1} in $D^{n-3} \times D_s$ discussed above (depending on $g: S^{n-1} \rightarrow O(n)$ etc.) and the standard trivialization of the normal bundle of $1/2$ in $[0, 1]$. As such it is homotopic to the standard trivialization, due to the fact that the composition of $g: S^{n-1} \rightarrow O(n)$ with the inclusion of $O(n)$ in $O(n+1)$ is nullhomotopic. To sum up, $\Phi^{\text{ad}}|_U$ is easy to describe and poor in distinguishing features.

Now let

$$(\psi_t: U \rightarrow D^{n-3} \times D_s \times [0, 1])_{t \in [0, 1]}$$

be any smooth isotopy of smooth codimension zero embeddings, with ψ_0 equal to the inclusion and each ψ_t avoiding the boundary of $D^{n-3} \times D_s \times [0, 1]$. Write $U_t := \psi_t(U)$ which we view as a neighborhood of $\psi_t(S_\delta^{n-1})$ in $D^{n-3} \times D_s \times [0, 1]$. In particular $U_0 = U$. Let σ_t be a smooth map from U_t to $\text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2) \setminus 0$

which is transverse to the rank 1 stratum, with transverse preimage $\psi_t(S_\delta^{n-1})$; we are assuming that σ_t depends smoothly on t and $\sigma_0 = \Phi^{\text{ad}}|_{U_0}$. Then it is easy to deform Φ by a homotopy, rel boundary of $D^{n-3} \times [0, 1]$, to a map Φ_1 whose adjoint Φ_1^{ad} is smooth and transverse to the rank 1 stratum, with transverse preimage of the rank 1 stratum equal to $\psi_1(S_\delta^{n-1})$, and agrees with σ_1 on a neighborhood of $\psi_1(S_\delta^{n-1})$ contained in U_1 .

Our strategy is to choose a deformation $(\psi_t, \sigma_t)_{t \in [0, 1]}$ as above in such a way that σ_1 integrates to a smooth map $\int \sigma_1 : U_1 \rightarrow \mathbb{R}^2$ which, fiberwise over D^{n-3} , satisfies the conditions on jets and multijets which we used to define $\mathcal{W}^?$. This is enough to ensure that the map Φ_1 lifts rel boundary to a map $\bar{\Phi}_1$ from $D^{n-3} \times [0, 1]$ to $\mathcal{W}^?$. That lift will be a nullhomotopy for the map $D^{n-3}/\partial \rightarrow \mathcal{W}^?$ determined by f . More precisely, we find it convenient to produce a map

$$\Phi_2 : D^{n-3} \times [0, 1] \rightarrow \mathcal{W}$$

with a smooth adjoint $\Phi_2^{\text{ad}} : D^{n-3} \times D_s \times [0, 1] \rightarrow \mathbb{R}^2$ such that the fiberwise derivative

$$d\Phi_2^{\text{ad}} : D^{n-3} \times D_s \times [0, 1] \rightarrow \text{hom}(\mathbb{R}^{n+2}, \mathbb{R}^2) \setminus 0$$

is transverse to the rank 1 stratum, and its restriction σ_1 to a tubular neighborhood U_1 of the transverse preimage of the rank 1 stratum can be related to σ_0 on U_0 by a deformation as discussed. We do not claim that Φ_2 is a candidate for $\bar{\Phi}_1$.

Such a map Φ_2 is easy to find. We choose a path $\gamma : [0, 1] \rightarrow \mathcal{W}$, also written $(\gamma(t))_{t \in [0, 1]}$, such that

- $\gamma(\delta)$ has exactly one singular point which is the center of D_s ;
- that singular point has type *lips*;
- the germ of the path at $t = \delta$ and the center of D_s is a universal unfolding of the singular point of $\gamma(\delta)$;
- the $\gamma(t)$ for $t > \delta$ are singularity free, and $\gamma(1)$ is the base point of \mathcal{W} ;
- the $\gamma(t)$ for $t < \delta$ are all in the same (global) left-right equivalence class.

Then the singularity set in the source of each $\gamma(t)$ for $t < \delta$ is smooth circle in D_s , of which two points are singularities of type *cusp* while the rest are of type *fold*. The singular set in the target is a standard lips picture with two fold lines and two cusps. We define Φ_2 by taking a point in $D^{n-2} \subset D^{n-3} \times [0, 1]$ with norm t to $\gamma(t)$, and all remaining points to the base point of \mathcal{W} . We can also ensure that the adjoint of Φ_2 is smooth by imposing conditions on the path γ . The details are left to the reader.

Remark 6.4. In the proof above, the choice of $\gamma(\delta)$ is not unique. For a start, there are several left-right equivalence classes of smooth map germs $(\mathbb{R}^{n+2}, 0) \rightarrow (\mathbb{R}^2, 0)$ entitled to the name *lips*. We can make representatives for all of them by choosing a fixed smooth map germ $u : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ of type *lips* such that the linear map $du(0)$ has image $\mathbb{R} \times 0 \subset \mathbb{R}^2$, and a nondegenerate symmetric form q in n variables, and defining a new germ (of type *lips*) by

$$v : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^2 \quad ; \quad v(z_1, \dots, z_n, x, y) = (u_1(x, y), u_2(x, y) + q(z_1, \dots, z_n)) .$$

In choosing $\gamma(\delta)$ for the proof above, start with a good choice of u (orientation preserving away from the singular point) and continue with a good choice of q (only the parity of the index matters).

Remark 6.5. In the proof above, it was necessary to deform Φ before attempting to lift to $\mathcal{W}^?$. We saw that the fiberwise singularity set of the adjoint Φ^{ad} is

$$S_\delta^{n-1} \times 1/2$$

which projects to the parameter space $D^{n-3} \times [0, 1]$ by a constant map. In integrable circumstances, the projection could not be constant because its fibers can have dimension at most 1.

A. INTERPOLATION AND TAMENESS

In this Appendix we give proofs of theorem 2.3 and theorem 4.11.

Let Q_n be the ring of polynomial maps $\mathbb{R}^N \rightarrow \mathbb{R}$ of degree $\leq n$.

Lemma A.1. *For fixed $d > 0$, there exists n such that Q_n is transverse to all codimension d ideals of the space $C^\infty(\mathbb{R}^N, \mathbb{R})$.*

Proof. See for example Glaeser [7] or Vokrinek [27]. The main point is that for every codimension d ideal \mathfrak{p} there exist a finite set $S \subset \mathbb{R}^N$ and a function v from S to $\{1, \dots, d\}$ such that

$$(A.1) \quad \prod_{x \in S} \mathfrak{m}_x^{v(x)} \subset \mathfrak{p} \subset \prod_{x \in S} \mathfrak{m}_x .$$

Here \mathfrak{m}_x is the maximal ideal $\{f \in C^\infty(\mathbb{R}^N, \mathbb{R}) \mid f(x) = 0\}$. \square

For a codimension d ideal $\mathfrak{p} \subset C^\infty(\mathbb{R}^N, \mathbb{R})$, sandwiched as in (A.1), the set S will be called the *spectrum* of \mathfrak{p} .

Lemma A.2. *Let $n, m, d \in \mathbb{N}$ be fixed. The space of m -dimensional affine subspaces of Q_n which are transverse to all codimension d ideals of $C^\infty(\mathbb{R}^N, \mathbb{R})$ with spectrum contained in L is an open set in the space of m -dimensional affine subspaces of Q_n .*

Proof. By [27], the space of codimension d ideals with spectrum contained in L is a compact Hausdorff space X , and there is a canonical vector bundle E over X , with d -dimensional fibers, so that the fiber over $\mathfrak{p} \in X$ is equal to the quotient ring $C^\infty(\mathbb{R}^N, \mathbb{R})/\mathfrak{p}$. An m -dimensional affine subspace A of Q_n is transverse to all codimension d ideals of $C^\infty(\mathbb{R}^N, \mathbb{R})$ with spectrum contained in L if and only if the canonical vector bundle map $X \times A \rightarrow E$ is fiberwise epimorphic. Hence the space of these affine subspaces is an open subset of the space (product of Q_n itself and a Grassmannian) of all m -dimensional affine subspaces of Q_n . \square

Lemma A.3. *Let $n, m, d \in \mathbb{N}$ be fixed. The space of m -dimensional affine subspaces of Q_n^k which are transverse to $\mathfrak{p}_1 \times \mathfrak{p}_2 \times \dots \times \mathfrak{p}_k$ for all k -tuples $(\mathfrak{p}_1, \dots, \mathfrak{p}_k)$ of codimension d ideals of $C^\infty(\mathbb{R}^N, \mathbb{R})$, all with spectrum contained in L , is an open set in the space of m -dimensional affine subspaces of Q_n^k .*

Proof. Same as for the lemma A.2 above. \square

Define $g_L: L \rightarrow \mathbb{R}$ by $g_L = \sum_{i=1}^s f_i^r$, where f_1, \dots, f_s are polynomial functions on \mathbb{R}^N which generate the ideal defining ∂L as a nonsingular algebraic subset of \mathbb{R}^N . As r is even, we have $g_L^{-1}(0) = \partial L$. The $(r+1)$ -th partial derivative of g_L in the direction tangential to L and normal to ∂L is everywhere nonzero on ∂L .

Lemma A.4. *Let $i \in \mathbb{N}$ and $d = i \dim(P)$. Suppose that $K \subset Q_n^k$ is a finite dimensional linear subspace, transverse to $\mathfrak{p}_1 \times \cdots \times \mathfrak{p}_k$ for all tuples $(\mathfrak{p}_1, \dots, \mathfrak{p}_k)$ of codimension $\leq d$ ideals of $C^\infty(\mathbb{R}^N, \mathbb{R})$, all with spectrum contained in L . Then $\varphi + g_L \cdot K \subset Sm$, and for every subset T of $L \setminus \partial L$ with $|T| \leq i$, the projection*

$$\varphi + g_L \cdot K \longrightarrow \prod_{x \in T} J_x^r(L, \mathbb{R}^k) \cong P^T$$

is onto.

Proof. For $T \subset L \setminus \partial L$ with $|T| \leq i$, let $\mathfrak{p}_1 = \mathfrak{p}_2 = \cdots = \mathfrak{p}_k$ be the ideal of $C^\infty(\mathbb{R}^N, \mathbb{R})$ consisting of all functions f whose jet $j_x^r f$ vanishes for all $x \in T$. This has codimension $\leq d$. By assumption, K is transverse to $\mathfrak{p}_1 \times \cdots \times \mathfrak{p}_k$. It follows that the projection

$$K \longrightarrow \prod_{x \in T} J_x^r(L, \mathbb{R}^k)$$

is onto. Since g_L is nonzero at all points of T , we deduce that

$$\varphi + g_L \cdot K \longrightarrow \prod_{x \in T} J_x^r(L, \mathbb{R}^k)$$

is onto. □

We will eventually construct the affine spaces A_i of theorem 2.3 in the form $A_i = \varphi + g_L \cdot K_i$ for suitable $n \gg 0$ (depending on i) and finite dimensional linear subspaces $K_i \subset Q_n^k$. The three lemmas above show, broadly speaking, that the interpolation property 2.3 (iv) restricts our choice of K_i to a *non-empty open* collection of finite-dimensional linear spaces. In the following we aim to show that the tameness property 2.3 (iii) restricts our choice of K_i to a *dense* collection.

Fix a linear space K of polynomial maps $\mathbb{R}^N \rightarrow \mathbb{R}^k$, of finite dimension m_0 . Suppose that K is *tame* in the following sense: there exists $\alpha \in \mathbb{N}$ such that for every $f \in K$, the number of inadmissible points of $\varphi + g_L \cdot f$ on L is $< \alpha$. Fix such an α . (A large α is preferred; the reasons are given below.) Next, choose n sufficiently large so that the projection

$$Q_n \rightarrow \prod_{x \in T} J_x^r(L, \mathbb{R}^k)$$

is onto for every finite subset $T \subset L$ of cardinality at most α , and so that $K \subset Q_n^k$. By lemma A.1, such an n exists. We fix $m_1 > m_0$ and turn our attention to the Grassmannian Y of m_1 -dimensional linear subspaces of Q_n^k containing K . Let

$$C_K \subset Y$$

be the semialgebraic subset consisting of all K' which are *wild* (as opposed to tame) in the sense that there is $f \in K'$ such that $\varphi + g_L \cdot f$ has $\geq \alpha$ inadmissible points on L .

Lemma A.5. *If $2\alpha > m_1$, then C_K is nowhere dense in Y .*

Proof. Let EY be the tautological vector bundle on Y , with fiber K' over the point $K' \in Y$. Let Z be the set of all (K', f, S) where $K' \in Y$, $f \in K'$ and $S \subset L$ has cardinality α and consists of inadmissible points for $\varphi + g_L \cdot f$ (so that $S \cap \partial L = \emptyset$). Then

$$Z \subset EY \times L^\alpha$$

as a semialgebraic subset. By the interpolation property which we are assuming for Q_n , and by our assumption on \mathfrak{X} , the codimension of this semialgebraic subset is at least $\alpha \cdot (\ell + 2)$ (see remark A.6). The image of Z under the projection to Y therefore still has codimension at least $2\alpha - m_1$ everywhere. That image is precisely C_K . Therefore the codimension of C_K in Y is at least 1 if $2\alpha > m_1$. \square

Now we construct the finite dimensional linear spaces K_i , and the affine spaces $A_i = \varphi + g_L \cdot K_i$ of theorem 2.3. Suppose per induction that K_i has already been constructed for a specific $i \geq 0$, has dimension m_0 and consists of polynomial maps from \mathbb{R}^N to \mathbb{R}^k . We also assume that the following condition (λ_i) is satisfied: for every k -tuple $f = (f_1, \dots, f_k)$ of monomials of degree $\leq i$ in N variables, there exists a map in K_i whose distance from f in the C^i topology (on the space of smooth maps from L to \mathbb{R}^k) is less than 2^{-i} .

Choose $s \in \mathbb{N}$ large enough so that K_i is properly contained in Q_s^k and Q_s is transverse to all codimension $\leq d$ ideals of the space $C^\infty(\mathbb{R}^N, \mathbb{R})$, where

$$d = (i + 1) \dim(P) .$$

Let $m_1 = \dim(Q_s^k)$. Choose $\alpha > m_1/2$ and choose n sufficiently large so that the projection

$$Q_n \rightarrow \prod_{x \in T} J_x^r(L, \mathbb{R}^k)$$

is onto for every finite subset $T \subset L$ of cardinality at most α . Also, arrange $n > s$ and $s > i$. By lemmas A.2 and A.5, there exists a linear subspace K_{i+1} of Q_n^k of dimension m_1 and as close to Q_s^k as we might wish, with the following properties:

- $K_i \subset K_{i+1}$;
- K_{i+1} is transverse to $\mathfrak{p}_1 \times \dots \times \mathfrak{p}_k$ for all tuples $(\mathfrak{p}_1, \dots, \mathfrak{p}_k)$ of codimension $\leq d$ ideals of $C^\infty(\mathbb{R}^N, \mathbb{R})$, all with spectrum contained in L , where

$$d = (i + 1) \dim(P) ;$$

- any set of inadmissible points, alias bad event, (in L) for any $f \in A_{i+1} := \varphi + g_L \cdot K_{i+1}$ has cardinality $< \alpha$.

Then the tameness condition for $A_{i+1} = \varphi + g_L \cdot K_{i+1}$ is satisfied. The interpolation condition is also satisfied by lemma A.4. Since K_{i+1} can be as close as we wish to Q_s , and $s > i$, we can choose it so that condition (λ_{i+1}) is satisfied. This completes the induction. Because condition (λ_i) is satisfied by K_i , for all i , the union $\bigcup_i A_i$ is dense in Sm .

Remark A.6. Suppose that A_i is a finite dimensional affine subspace of Sm which has the interpolation property (iv) of theorem 2.3. Let $T = \{1, 2, \dots, i\}$ and let

$$Z \subset A_i \times \text{emb}(T, L \setminus \partial L)$$

consist of all (f, e) such that $e(T)$ is a bad event for f . Suppose we fix a point $(f, e) \in Z$ and choose a smooth trivialization of the tangent bundle TL near $e(T)$, not necessarily algebraic. We can then use this to trivialize the jet bundle near $e(T)$ and construct a germ of maps

$$A_i \times \text{emb}(T, L \setminus \partial L) \longrightarrow \prod_{x \in T} J_x^r(\mathbb{R}^\ell, \mathbb{R}^k) \cong P^T$$

(near (f, e)) by mapping an element (g, b) to the multijet of g at $b(T)$. Then the germ of Z near (f, e) is the preimage of $(P \setminus \mathfrak{X})^T$ under this map germ. On the other

hand, by the interpolation property, the map germ is a submersion. Therefore the codimension of the germ of Z near (f, e) , as a germ of manifold stratified subsets of $A_i \times \text{emb}(T, L \setminus \partial L)$, is \geq the codimension of $(P \setminus \mathfrak{X})^T$ in P^T , which is $\geq i(\ell + 2)$. Similarly, in the situation of lemma A.5, we have

$$Z \subset EY \times L^\alpha.$$

Given $(K', f, S) \in Z$, choose a trivialization of TL near $f(S)$ and, as above, use this to construct a submersion germ

$$(EY \times L^\alpha, (K', f, S)) \longrightarrow \prod_{x \in S} J_x^r(\mathbb{R}^\ell, \mathbb{R}^k) \cong P^S$$

such that the germ of Z near (K', f, S) is the preimage of $(P \setminus \mathfrak{X})^S$ under this map germ. Therefore the codimension of the germ of Z near (K', f, S) is $\geq \alpha(\ell + 2)$.

Now we prove theorem 4.11. We will eventually construct the affine spaces A_i of theorem 2.3 in the form $A_i = \varphi + g_L \cdot K_i$ for suitable $n \gg 0$ (depending on i) and finite dimensional linear subspaces $K_i \subset Q_n^k$. The three lemmas above show, broadly speaking, that the interpolation property 2.3 (iv) restricts our choice of K_i to a *non-empty open* collection of finite-dimensional linear spaces. In the following we aim to show that the tameness property 2.3 (iii) restricts our choice of K_i to a *dense* collection.

Fix a linear space K of polynomial maps $\mathbb{R}^N \rightarrow \mathbb{R}^k$, of finite dimension m_0 . Suppose that K is *tame* in the following sense: there exists $\alpha \in \mathbb{N}$ such that for every $f \in K$, the cardinality of any bad event for $\varphi + g_L \cdot f$ on L is $< \alpha$. Fix such an α . Next, choose n sufficiently large so that the projection

$$Q_n \rightarrow \prod_{x \in T} J_x^r(L, \mathbb{R}^k)$$

is onto for every finite subset $T \subset L$ of cardinality at most $\alpha + \zeta - 1$ where ζ is the upper bound of definition 4.5, and so that $K \subset Q_n^k$. By lemma A.1, such an n exists. We fix $m_1 > m_0$ and turn our attention to the Grassmannian Y of m_1 -dimensional linear subspaces of Q_n^k containing K . Let

$$C_K \subset Y$$

be the semialgebraic subset consisting of all K' which are *wild* in the sense that there is $f \in K'$ such that $\varphi + g_L \cdot f$ has a bad event of cardinality $\geq \alpha$ on L .

Lemma A.7. *If $2\alpha > m_1$, then C_K is nowhere dense in Y .*

Proof. Let EY be the tautological vector bundle on Y , with fiber K' over the point $K' \in Y$. Let Z be the set of all (K', f, S) where $K' \in Y$ and $f \in K'$ and $S \subset L$ a bad event for $\varphi + g_L \cdot f$ whose cardinality is between α and $\alpha + \zeta - 1$. Then

$$Z \subset EY \times L^\alpha$$

as a semialgebraic subset. From here onwards we proceed as in the proof of lemma A.5. \square

Now we construct the finite dimensional linear spaces K_i , and the affine spaces $A_i = \varphi + g_L \cdot K_i$ of theorem 4.11. Suppose per induction that K_i has already been constructed for a specific $i \geq 0$, has dimension m_0 and consists of polynomial maps from \mathbb{R}^N to \mathbb{R}^k . We also assume that the following condition (λ_i) is satisfied: for

every k -tuple $f = (f_1, \dots, f_k)$ of monomials of degree $\leq i$ in N variables, there exists a map in K_i whose distance from f in the C^i topology (on the space of smooth maps from L to \mathbb{R}^k) is less than 2^{-i} .

Choose $s \in \mathbb{N}$ large enough so that K_i is properly contained in Q_s^k and Q_s is transverse to all codimension $\leq d$ ideals of the space $C^\infty(\mathbb{R}^N, \mathbb{R})$, where

$$d = (i + 1) \dim(P) .$$

Let $m_1 = \dim(Q_s^k)$. Choose $\alpha > m_1/2$ and choose n sufficiently large so that the projection

$$Q_n \rightarrow \prod_{x \in T} J_x^r(L, \mathbb{R}^k)$$

is onto for every finite subset $T \subset L$ of cardinality at most $\alpha + \zeta - 1$. Also, arrange $n > s$ and $s > i$. By lemmas A.2 and A.7, there exists a linear subspace K_{i+1} of Q_n^k of dimension m_1 and as close to Q_s^k as we might wish, with the following properties:

- $K_i \subset K_{i+1}$;
- K_{i+1} is transverse to $\mathfrak{p}_1 \times \dots \times \mathfrak{p}_k$ for all tuples $(\mathfrak{p}_1, \dots, \mathfrak{p}_k)$ of codimension $\leq d$ ideals of $C^\infty(\mathbb{R}^N, \mathbb{R})$, all with spectrum contained in L , where

$$d = (i + 1) \dim(P) ;$$

- any bad event (in L) for any $f \in A_{i+1} := \varphi + g_L \cdot K_{i+1}$ has cardinality $< \alpha$.

Then the tameness condition for $A_{i+1} = \varphi + g_L \cdot K_{i+1}$ is satisfied. The interpolation condition is also satisfied by lemma A.4. Since K_{i+1} can be as close as we wish to Q_s , and $s > i$, we can choose it so that condition (λ_{i+1}) is satisfied. This completes the induction. Because condition (λ_i) is satisfied by K_i , for all i , the union $\bigcup_i A_i$ is dense in Sm .

B. TOPOLOGY OF BADNESS

Fix a positive integer α and let X be the space of nonempty subsets of \mathbb{R}^N of cardinality at most α , equipped with the Hausdorff metric. Let Y be the space of all maps from $\{1, 2, 3, \dots, \alpha\}$ to \mathbb{R}^N , topologized as the α -fold power of \mathbb{R}^N . There is a surjective map $q: Y \rightarrow X$ taking $f \in Y$ to its image $\text{im}(f)$.

Lemma B.1. *The map q is an identification map.*

Proof. For fixed $f \in Y$ and $\varepsilon > 0$, the set of $g \in Y$ whose image $\text{im}(g)$ has Hausdorff distance $< \varepsilon$ from $\text{im}(f)$ is clearly open in Y . This means that q is continuous. Next, let $S \in X$ be fixed and let $U \subset X$ be such that $S \in U$. Suppose that U is not a neighborhood of S . We need to show that $q^{-1}(U)$ is not a neighborhood of $q^{-1}(S)$. Since U is not a neighborhood of S , there exists a sequence $(S_i)_{i \in \mathbb{N}}$ in X converging to S such that $S_i \notin U$ for all i . Choose $\varepsilon > 0$ such that the ε -neighborhoods of the $s \in S$ are pairwise disjoint in \mathbb{R}^N . Each S_i determines a function $m_i: S \rightarrow \mathbb{N}$, where $m_i(s)$ is the number of elements of S_i which have distance $< \varepsilon$ from $s \in S$. As $\sum_{t \in S} m_i(t) \leq \alpha$, there are only finitely many possibilities for the m_i . Hence the sequence $(S_i)_{i \in \mathbb{N}}$ has a subsequence $(S_{i_r})_{r \in \mathbb{N}}$ such that m_{i_r} is the same for all r . We rename: $T_r = S_{i_r}$. Now it is easy to lift the sequence $(T_r)_{r \in \mathbb{N}}$ to a convergent sequence $(f_r)_{r \in \mathbb{N}}$ in Y , with a limit which we call f . Then $f \in q^{-1}(S)$ but $f_r \notin q^{-1}(U)$, so that $q^{-1}(U)$ is not a neighborhood of $q^{-1}(S)$. \square

The equivalence relation on Y which has $f \sim g$ if and only if $\text{im}(f) = \text{im}(g)$ is fairly easy to understand. In particular the following is easy.

Lemma B.2. *There is a triangulation of Y which agrees with the linear structure (every simplex is the convex hull in Y of its vertex set) and which descends to a triangulation of the quotient Y/\sim .*

Next let A be a finite dimensional affine subspace of Sm (as in section 2 or section 4) which consists of polynomial maps. Assume that A is tame in the sense that the cardinality of any bad event of any $f \in A$ is not greater than α . Let $B \subset A$ be the subset of all $f \in A$ which have a bad event (see section 2 or definition 4.9). Let Λ be the poset of all pairs (f, S) such that $f \in B$ and $S \subset L \setminus \partial L$ is a bad event for f . The order relation has $(f, S) \leq (g, T)$ if and only if $f = g$ and $S \subset T$. We can write

$$\Lambda \subset B \times X.$$

Corollary B.3. *Λ is an ENR and the projection $\Lambda \rightarrow B$ is proper.*

Proof. By [14, Ch. VI, thm. 1.2], to show that Λ is an ENR and closed in $B \times X$, it is enough to show that $\Lambda \cap (B \times Q)$ is a closed ENR in $B \times Q$ for every simplex $Q \subset X$. By lemma B.2 we can lift Q to a simplex \bar{Q} in the vector space Y . It remains to show that the preimage of $\Lambda \subset B \times X$ under the composite map

$$B \times \bar{Q} \xrightarrow{\text{incl.}} B \times Y \xrightarrow{\text{id} \times q} B \times X$$

is an ENR. That preimage is clearly a closed semialgebraic subset of $A \times Y$. For the properness statement we introduce Z , the space of subsets of L of cardinality $\leq \alpha$. This is a quotient of L^α . Hence it is compact. The inclusion of RB in $B \times X$ factors as

$$RB \longrightarrow B \times Z \longrightarrow B \times X.$$

By the above, RB is closed in $B \times Z$ since it is closed in $B \times X$. The projection $B \times Z \rightarrow B$ is proper since Z is compact. Therefore its restriction to RB is also proper. \square

Corollary B.4. *For $s \geq 0$, let $N_s \Lambda$ be the set of all order-preserving maps from $\{0, 1, \dots, s\}$ to Λ . Then $N_s \Lambda$ is an ENR and the projection $N_s \Lambda \rightarrow B$ is proper.*

Proof. This is similar to the previous proof. We do the case $s = 1$. Let Q_0 and Q_1 be simplices in X with lifts \bar{Q}_0 and \bar{Q}_1 to Y . We need to show that the preimage in

$$(B \times \bar{Q}_0) \times (B \times \bar{Q}_1)$$

of the order relation in $\Lambda \times \Lambda \subset (B \times X) \times (B \times X)$ is a closed sub-ENR. That preimage is a closed semialgebraic subset of $(A \times Y) \times (A \times Y)$. Hence it is an ENR. We conclude as before that $N_1 \Lambda$ is an ENR, closed in $\Lambda \times \Lambda$. The inclusion of $N_1 \Lambda$ in $\Lambda \times \Lambda$ factors as

$$N_1 \Lambda \longrightarrow B \times Z \times Z \longrightarrow \Lambda \times \Lambda,$$

with Z as in the previous proof. Hence $N_1 \Lambda$ is closed in $B \times Z \times Z$. Since the projection from $B \times Z \times Z$ to B is proper, its restriction to $N_1 \Lambda$ is also proper. \square

Finally we consider Λ as a topological poset, with a classifying space which we call RB to keep the analogy with section 2. It is the geometric realization of a simplicial

space $s \mapsto N_s \Lambda$. Each $N_s \Lambda$ is an ENR and all simplicial operators $N_s \Lambda \rightarrow N_t \Lambda$ are proper. Because the length of any chain of non-identity morphisms in Λ is bounded by α , every element of $N_s \Lambda$ for $s > \alpha$ is in the image of some degeneracy operator. These facts together with [14, Ch. VI, thm 1.2] imply the following.

Proposition B.5. *The classifying space RB of Λ is an ENR and the projection $RB \rightarrow B$ is proper.*

C. EXCISION IN LOCALLY FINITE HOMOLOGY

Let A be an ENR, and B a closed subset of A which is also an ENR.

Proposition C.1. *There is an isomorphism $H_*^{\text{lf}}(A, B) \cong H_*^{\text{lf}}(A \setminus B)$.*

Proof. This is well known. It can be proved using (unreduced) Steenrod homology, a homology theory H_*^{st} for pairs of compact metric spaces [23, 24, 18, 6]. Milnor showed [18] that H_*^{st} satisfies, in addition to the seven Eilenberg-Steenrod axioms for a homology theory, two further axioms numbered eight and nine. Axiom eight is a wedge axiom for a type of infinite wedge, while axiom nine states that for a pair (X, Y) of compact metric spaces, the identification map $(X, Y) \rightarrow (X/Y, \star)$ induces an isomorphism in Steenrod homology. These axioms imply that for a pair of finite dimensional locally finite CW-spaces (A, B) with one-point compactifications A^ω and B^ω , the Steenrod homology of the pair (A^ω, B^ω) is isomorphic to the locally finite homology of the pair (A, B) . Indeed the spectral sequence in Steenrod homology determined by the skeleton filtration of A relative to B collapses (by the axioms) at the E^2 page, which gives the isomorphism. It follows that $H_*^{\text{st}}(A^\omega, B^\omega)$ is also isomorphic to the locally finite homology of (A, B) when (A, B) is a pair of ENRs with B closed in A . But by axiom number nine, $H_*^{\text{st}}(A^\omega, B^\omega)$ is also isomorphic to $H_*^{\text{st}}(A/B, \star)$ which in turn is isomorphic to the locally finite homology of $A \setminus B$. \square

Corollary C.2. *Let B be a closed ENR in \mathbb{R}^d . Then*

$$\tilde{H}^*(\mathbb{R}^d \setminus B) \cong H_{d-*}^{\text{lf}}(B)$$

where \tilde{H}^* is reduced cohomology.

Proof. The long exact sequence in locally finite homology of the pair (\mathbb{R}^d, B) shows

$$H_{d-s-1}^{\text{lf}}(B) \cong H_{d-s}^{\text{lf}}(\mathbb{R}^d, B)$$

for $s > 0$, because $H_*^{\text{lf}}(\mathbb{R}^d)$ is isomorphic to $H^{d-*}(\mathbb{R}^d)$ by Poincaré duality for noncompact manifolds. By proposition C.1 and another application of Poincaré duality, we have

$$H_{d-s}^{\text{lf}}(\mathbb{R}^d, B) \cong H_{d-s}^{\text{lf}}(\mathbb{R}^d \setminus B) \cong H^s(\mathbb{R}^d \setminus B) .$$

This isomorphism extends to a map of long exact sequences

$$\begin{array}{ccc}
\begin{array}{c} \vdots \\ \uparrow \\ H_{d-s-1}^{\ell f}(B) \end{array} & \longrightarrow & \begin{array}{c} \vdots \\ \uparrow \\ H^{s+1}(\mathbb{R}^d, \mathbb{R}^d \setminus B) \end{array} \\
\uparrow & & \uparrow \\
H_{d-s}^{\ell f}(\mathbb{R}^d, B) & \longrightarrow & H^s(\mathbb{R}^d \setminus B) \\
\uparrow & & \uparrow \\
H_{d-s}^{\ell f}(\mathbb{R}^d) & \longrightarrow & H^s(\mathbb{R}^d) \\
\uparrow & & \uparrow \\
H_{d-s}^{\ell f}(B) & \longrightarrow & H^s(\mathbb{R}^d, \mathbb{R}^d \setminus B) \\
\begin{array}{c} \vdots \\ \uparrow \end{array} & & \begin{array}{c} \vdots \\ \uparrow \end{array}
\end{array}$$

which, by the five lemma, is also an isomorphism. In particular

$$H_{d-s-1}^{\ell f}(B) \cong H^{s+1}(\mathbb{R}^d, \mathbb{R}^d \setminus B) \cong H^s(\mathbb{R}^d \setminus B) . \quad \square$$

Example C.3. Something like the ENR condition in corollary C.2 is necessary. The set $B = \{n^{-1} \in \mathbb{R} \mid n = 1, 2, 3, \dots\} \cup \{0\}$ is closed in \mathbb{R} , but it is not an ENR. It is compact, and so

$$H_0^{\ell f}(B) = H_0(B) \cong \bigoplus_{s \in B} \mathbb{Z} .$$

By contrast, $H^0(\mathbb{R} \setminus B) \cong \prod_{s \in B} \mathbb{Z}$. The two groups are not isomorphic since one is uncountable and the other is countable.

Remark C.4. There are situations where we need to promote proposition C.1 to a statement at the chain level. Therefore we shall briefly describe a chain map of locally finite singular chain complexes

$$(C.1) \quad C_*^{\ell f}(A, B) \longrightarrow C_*^{\ell f}(A \setminus B),$$

defined when (A, B) is a pair of locally compact spaces, with B closed in A . By $C_*^{\ell f}(A, B)$ we mean the quotient $C_*^{\ell f}(A)/C_*^{\ell f}(B)$, and we note that the inclusion $C_*^{\ell f}(B) \rightarrow C_*^{\ell f}(A)$ is a cofibration, i.e., split injective in each degree. If A and B are ENRs, then (C.1) induces an isomorphism in homology, the isomorphism of proposition C.1.

The idea is simple. Given any singular simplex $f: \Delta^q \rightarrow A$, we have the closed set $f^{-1}(B) \subset \Delta^q$. For $p \geq 0$ let S_p be the set of q -simplices in the p -th barycentric subdivision of Δ^q which have empty intersection with $f^{-1}(B)$ and which are *not* contained in a simplex of the $(p-1)$ -th subdivision which also has empty intersection with $f^{-1}(B)$. Then the formal sum

$$\sum_{p \geq 0} \sum_{\sigma \in S_p} \sigma$$

is a locally finite q -chain in $A \setminus B$. By mapping the q -simplex f to that locally finite q -chain, we obtain a chain map of the form (C.1). It has the following naturality property: Given locally compact A and closed subsets B_0 and B_1 with $B_0 \subset B_1$,

there is a commutative diagram

$$\begin{array}{ccc} C_*^{\text{eff}}(A, B_0) & \longrightarrow & C_*^{\text{eff}}(A \setminus B_0) \\ \downarrow & & \downarrow \\ C_*^{\text{eff}}(A, B_1) & \longrightarrow & C_*^{\text{eff}}(A \setminus B_1) \end{array}$$

where the horizontal arrows are of the form (C.1) and the right-hand vertical arrow is also of that type (note that $A \setminus B_1 = (A \setminus B_0) \setminus (B_1 \setminus B_0)$).

In the case where $A = \mathbb{R}^d$, the chain map (C.1) fits (by naturality) into a commutative diagram of chain maps

$$\begin{array}{ccccc} C_*^{\text{eff}}(\mathbb{R}^d) & \xrightarrow{=} & C_*^{\text{eff}}(\mathbb{R}^d) & \xleftarrow{\cong} & C^{d-*}(\mathbb{R}^d) \\ \downarrow & & \downarrow & & \downarrow \\ C_*^{\text{eff}}(\mathbb{R}^d, B) & \xrightarrow{\cong} & C_*^{\text{eff}}(\mathbb{R}^d \setminus B) & \xleftarrow{\cong} & C^{d-*}(\mathbb{R}^d \setminus B) \end{array}$$

Passing to the mapping cones of the vertical arrows, we obtain a chain of natural homotopy equivalences

$$C_*^{\text{eff}}(B) \longrightarrow \cdots \longleftarrow \tilde{C}^{d-*}(\mathbb{R}^d \setminus B)$$

where the tilde indicates a reduced cochain complex. (The reduced singular cochain complex of a space X is the mapping cone of the chain map $C^*(\star) \rightarrow C^*(X)$ induced by the unique map $X \rightarrow \star$.)

We state one more result about locally finite homology which does not depend on ENR assumptions.

Proposition C.5. *Let Y be any locally compact metrizable space such that the one-point compactification is also metrizable. Then $H_*^{\text{eff}}(Y \times (0, 1])$ is zero.*

Proof. Let X be the one-point compactification of $Y \times [0, 1]$ and let X' be the one-point compactification of $Y \times 0$. By axiom 8^{eff} in [18] for the Steenrod homology groups,

$$H_q^{\text{st}}(X, X') \cong H_q^{\text{st}}(X \setminus X') \cong H_q^{\text{eff}}(Y \times (0, 1]) .$$

Therefore it is enough to show that $H_q^{\text{st}}(X, X') = 0$. But this follows from the homotopy invariance property of the Steenrod homology groups (which is part of the seven Eilenberg-Steenrod axioms). \square

D. BOUNDARY CONDITIONS

Let φ be an admissible smooth map from L to \mathbb{R}^k . Here *admissible* means \mathfrak{X}_\bullet -admissible as in section 4; we recall that this generalizes \mathfrak{X} -admissible as in section 2. In our definition of the space $C^\infty(U, \mathbb{R}^k; \mathfrak{X}_\bullet, \varphi)$ there is a boundary condition (a): a smooth map $f: L \rightarrow \mathbb{R}^k$ which is everywhere admissible belongs to that space if its r -jet at every $x \in \partial L$ agrees with the r -jet of φ at x . This deviates slightly from the original definition [25, 26]. Vassiliev has the condition (b) that the full Taylor series of f at every $x \in \partial L$ agree with the full Taylor series of φ at x (in local coordinates at x). Condition (b) has some technical advantages, as we shall see in the proof of proposition D.3 below. Our weaker condition (a) also has some technical advantages. It is much easier to find polynomial maps which satisfy it, in the case where $L \subset \mathbb{R}^k$ is a semi-algebraic subset (details as in section 2).

Lemma D.1. *The two types of boundary conditions, (a) and (b), lead to weakly homotopy equivalent versions of $C^\infty(U, \mathbb{R}^k; \mathfrak{X}_\bullet, \varphi)$.*

Proof. Denote the two versions by X_a and X_b , respectively. The inclusion $X_b \rightarrow X_a$ is claimed to be a weak homotopy equivalence. To show this, let $K \subset X_a$ be any compact subset (for example, the image of a continuous map from a compact CW-space to X_a). Choose a compact collar Q on ∂L and identify it with $\partial L \times [0, 1]$, relative to $\partial L \cong L \times 0$. Choose a smooth function $\psi: [0, 1] \rightarrow [0, 1]$ so that $\psi(t) = 0$ for t close to 0 and $\psi(t) = 1$ for t close to 1. Define $\Psi: L \rightarrow [0, 1]$ so that $\Psi(x, t) = \psi(t)$ for (x, t) in $\partial L \times [0, 1] \cong Q \subset L$ and $\Psi(y) = 1$ for all $y \in L \setminus Q$. We ask whether, for $s \in [0, 1]$ and $f \in K$, the map

$$f_s := (1 - s)f + s(\Psi \cdot f + (1 - \Psi)\varphi)$$

from L to \mathbb{R}^k is everywhere admissible. If so, then $f \mapsto (f_s)_{s \in [0, 1]}$ defines a homotopy from the inclusion $K \rightarrow X_a$ to a map which lands in the subspace X_b . Moreover, if K is already contained in X_b , then the homotopy would not take it out of X_b . In short, if the formula for f_s is good, then the proof is complete.

Of course there is no guarantee that the formula is good as it stands. But we can easily improve on it. Let $Q^\varepsilon = \partial L \times [0, \varepsilon] \subset \partial L \times [0, 1] \cong Q$ and define $\Psi^\varepsilon: L \rightarrow [0, 1]$ by $\Psi^\varepsilon(x, t) = \Psi(x, t/\varepsilon)$ for $(x, t) \in Q^\varepsilon$ and $\Psi^\varepsilon(y) = 1$ for all $y \in L \setminus Q^\varepsilon$. A calculation which we leave to the reader shows that the above formula for f_s turns into a good formula if we replace Ψ by Ψ^ε for sufficiently small ε . This is based on the existence of a constant $c > 0$ such that for $f \in K$ and p with $0 \leq p \leq r$, all p -th partial derivatives of $f - \varphi$ at a point $(x, t) \in \partial L \times [0, 1] \cong Q \subset L$ are bounded in size by ct^{r+1-p} . \square

Remark D.2. Lemma D.1 justifies a statement made in section 2, namely, there is no loss of generality in assuming that r is even. We recall that $\mathfrak{X}_T \subset P^T$ for a finite nonempty set T , where P is the vector space of polynomial maps of degree $\leq r$ from \mathbb{R}^ℓ to \mathbb{R}^k . Let \mathfrak{Y}^T be the preimage of \mathfrak{X}_T under the projection $P_1^T \rightarrow P^T$, where P_1 is the vector space of polynomial maps of degree $\leq r + 1$ from \mathbb{R}^ℓ to \mathbb{R}^k . There is an inclusion

$$C^\infty(U, \mathbb{R}^k; \mathfrak{Y}_\bullet, \varphi) \longrightarrow C^\infty(U, \mathbb{R}^k; \mathfrak{X}_\bullet, \varphi) .$$

We are using type (a) boundary conditions in the sense of lemma D.1. If L has empty boundary, this inclusion is an identity. If ∂L is nonempty, it would still be an identity if type (b) boundary conditions were used; therefore it is a weak equivalence with type (a) boundary conditions.

Next, let φ_0 and φ_1 be two admissible smooth maps from L to \mathbb{R}^k . Here *admissible* means \mathfrak{X}_\bullet -admissible as in section 4; we recall that this generalizes \mathfrak{X} -admissible as in section 2.

Proposition D.3. *The contravariant functors F_0 and F_1 on $\mathcal{O}^t(L)$ defined by*

$$F_0(U) = C^\infty(U, \mathbb{R}^k; \mathfrak{X}_\bullet, \varphi_0) , \quad F_1(U) = C^\infty(U, \mathbb{R}^k; \mathfrak{X}_\bullet, \varphi_1)$$

are related by a chain of natural weak homotopy equivalences.

The proof is an elaboration of an argument given in [26]. As a preparation, we choose a compact collar Q on ∂L , so that $Q \subset L$ and $\partial L \subset Q \cong \partial L \times [0, 2]$. Let $\mathcal{O}^t(L; Q) \subset \mathcal{O}^t(L)$ be the full sub-poset consisting of the $U \in \mathcal{O}^t(L)$ which contain the collar Q .

Lemma D.4. *There is a natural weak homotopy equivalence from $F_1|\mathcal{O}^t(L; Q)$ to $F_0|\mathcal{O}^t(L; Q)$.*

Proof. We use Vassiliev's boundary condition (b) throughout, as in lemma D.1. The collar Q can be identified with $\partial L \times [0, 2]$, relative to $\partial L \cong \partial L \times 0$. Choose $\varepsilon > 0$ and a smooth embedding $\psi: [0, 2] \rightarrow [0, 2]$ which extends

- the map $t \mapsto t + 1$ on $[0, 2\varepsilon]$;
- the map $t \mapsto t + \frac{1}{2}$ on $[1, 1 + 2\varepsilon]$;
- the identity on $[2 - \varepsilon, 2]$.

Choose a smooth admissible map φ_2 from $\partial L \times [0, 1]$ to \mathbb{R}^k such that $\varphi_2(x, t) = \varphi_0(x, t)$ for $t \in [0, \varepsilon]$ and such that the map from $\partial L \times [0, 2]$ to \mathbb{R}^k defined by $(x, t) \mapsto \varphi_2(x, t)$ for $t \in [0, 1]$ and $(x, t) \mapsto \varphi_1(x, t - 1)$ for $t \in [1, 2]$ is smooth.

Given U in $\mathcal{O}^t(L)$ containing Q , and $f \in F_1(U)$, define $f^\# \in F_0(U)$ in such a way that $f^\#$ agrees with φ_2 on $\partial L \times [0, 1]$, with $f \circ (\text{id} \times \psi^{-1})$ on $\partial L \times [1, 2]$, and with f on the complement of $Q = \partial L \times [0, 2]$ in U . Then $f \mapsto f^\#$ defines a natural transformation ν_1 from $F_1|\mathcal{O}^t(L; Q)$ to $F_0|\mathcal{O}^t(L; Q)$. A natural transformation ν_0 in the opposite direction, from $F_0|\mathcal{O}^t(L; Q)$ to $F_1|\mathcal{O}^t(L; Q)$, can be defined by almost the same formula. Of course, φ_2 must be replaced by a map φ_3 from $\partial L \times [0, 1]$ to \mathbb{R}^k such that $\varphi_3(x, t) = \varphi_1(x, t)$ for $t \in [0, \varepsilon]$ and such that the map from $\partial L \times [0, 2]$ to \mathbb{R}^k defined by $(x, t) \mapsto \varphi_3(x, t)$ for $t \in [0, 1]$ and $(x, t) \mapsto \varphi_0(x, t - 1)$ for $t \in [1, 2]$ is smooth.

Now we need to show that the composition $\nu_1\nu_0: F_0|\mathcal{O}^t(L; Q) \rightarrow F_0|\mathcal{O}^t(L; Q)$ is a weak homotopy equivalence. For this purpose, fix δ so that $\varepsilon > \delta > 0$. Let F_0^δ be the subfunctor of F_0 such that $F_0^\delta(U)$ for $U \in \mathcal{O}^t(L)$ consists of the maps $f \in F_0(U)$ which agree with φ_0 on $\partial L \times [0, \delta] \subset Q$. We will show that $\nu_1\nu_0$ restricted to $F_0^\delta|\mathcal{O}^t(L; Q)$ is naturally homotopic to the inclusion of $F_0^\delta|\mathcal{O}^t(L; Q)$ in $F_0|\mathcal{O}^t(L; Q)$. We need a description of $\nu_1\nu_0$ restricted to $F_0^\delta|\mathcal{O}^t(L; Q)$. For U in $\mathcal{O}^t(L; Q)$ and $f \in F_0^\delta(U)$, the map $\nu_1\nu_0(f): L \rightarrow \mathbb{R}^k$ agrees

- on $\partial L \times [\frac{3}{2}, 2]$ with $f \circ (\text{id} \times \psi^{-2})$;
- on the complement of $\partial L \times [0, 2]$ with f ;
- on $\partial L \times [0, \frac{3}{2} + \delta]$ with a map φ_4 which does not depend on f , where

$$\begin{aligned} \varphi_4(x, t) &= \varphi_0(x, t) \text{ for } t \in [0, \delta], \\ \varphi_4(x, t) &= \varphi_0(x, t - \frac{3}{2}) \text{ for } t \in [\frac{3}{2}, \frac{3}{2} + \delta]. \end{aligned}$$

Now choose a diffeomorphism $\sigma: [0, 2] \rightarrow [0, 2]$ which maps $[0, \delta]$ diffeomorphically to $[0, \frac{3}{2} + \delta]$ and agrees with ψ^2 on $[\delta, 2]$. We also require that $\sigma(t) = t$ for t close to 0 and $\sigma(t) = t + \frac{3}{2}$ for t close to δ . Let $\bar{\sigma}: L \rightarrow L$ be the diffeomorphism defined by $\bar{\sigma}(x, t) = (x, \sigma(t))$ for $(x, t) \in \partial L \times [0, 2] \cong Q$ and $\bar{\sigma}(y) = y$ for all $y \in L \setminus Q$. Then the maps

$$f \mapsto \nu_1\nu_0(f), \quad f \mapsto \nu_1\nu_0(f) \circ \bar{\sigma}$$

on $F_0^\delta(U)$ are homotopic, as one can see by choosing an appropriate isotopy from σ to the identity (inducing an isotopy from $\bar{\sigma}$ to the identity). The map

$$(D.1) \quad f \mapsto \nu_1\nu_0(f) \circ \bar{\sigma}$$

has the following property: $\nu_1\nu_0(f) \circ \bar{\sigma}$ agrees with f outside $\partial L \times [0, \delta] \subset Q$ while on $\partial L \times [0, \delta]$ it agrees with a map φ_5 which does not at all depend on f . We now choose a path $\gamma = (\gamma_s: \partial L \times [0, \delta] \rightarrow \mathbb{R}^k)_{s \in [0, 1]}$ starting with $\gamma_0 = \varphi_5$ and ending with the restriction of φ_0 . We choose this so that each γ_s agrees with φ_0 near $\partial L \times \partial[0, \delta]$. The path induces another homotopy from (D.1) to the inclusion

$F_0^\delta(U) \rightarrow F_0(U)$. The existence of such a path is guaranteed again by the “large enough” property of \mathfrak{X}_\bullet . (Note that when we selected the maps φ_2 and φ_3 , we used the assumption that the admissible maps form a dense subspace in the space of all maps satisfying the relevant boundary conditions; but in selecting the path γ we used the assumption that paths of admissible maps form a dense subspace in the space of paths of all maps satisfying the relevant boundary conditions.)

Now, for $U \in \mathcal{O}^t(L; Q)$, let K be any compact subset of $F_0(U)$. Then, as in the proof of lemma D.1, it is easy to deform K into a subspace of the form $F_0^\delta(U)$, for some $\delta > 0$. This means that $\nu_1\nu_0$ is homotopic to the identity on K . While this does not show that $\nu_1\nu_0$ is homotopic to the identity, it is enough to show that $\nu_1\nu_0$ is a weak homotopy equivalence, as claimed. \square

Sketch proof of proposition D.3. Given $Q \cong \partial L \times [0, 2]$ as in the proof of lemma D.1, we define Q^ε to be the part of Q corresponding to $\partial L \times [0, 2\varepsilon]$. For $U \in \mathcal{O}^t(L)$ we let

$$\mathbb{N}_U = \{n \in \mathbb{N} \mid Q^{1/n} \subset U\}.$$

Let $B\mathbb{N}_U$ be the classifying space of \mathbb{N}_U as an ordered set. This is clearly contractible. From the proof of lemma D.4 we have a natural map

$$B\mathbb{N}_U \times F_0(U) \longrightarrow F_1(U)$$

which is a weak homotopy equivalence for every U . In addition we have the natural projection $B\mathbb{N}_U \times F_0(U) \rightarrow F_0(U)$ which is a homotopy equivalence. These two constitute our chain of natural weak homotopy equivalences. \square

E. POLYNOMIAL FUNCTORS, ANALYTIC FUNCTORS AND DUALITY

Proposition E.1. *Let F be a good contravariant functor from $\mathcal{O}^t(L)$ to chain complexes of abelian groups. Suppose that $F(U)$ is homotopy equivalent to a chain complex of finitely generated free abelian groups, bounded below, for every U in $\mathcal{O}^t(L)$. Then the following are equivalent:*

- (1) *F is polynomial of degree $\leq p$;*
- (2) *The covariant functor DF defined by $DF(U) = \text{hom}(F(U), \mathbb{Z})$ is polynomial of degree $\leq p$.*

Proof. We recall what it means for F to be *good*: it means that for $U_0 \leq U_1$ in $\mathcal{O}^t(L)$ such that the inclusion $U_0 \rightarrow U_1$ is an isotopy equivalence (relative to ∂L), the induced map $F(U_1) \rightarrow F(U_0)$ induces an isomorphism in homology. With our assumptions, where $F(U_1)$ and $F(U_0)$ are chain complexes of free abelian groups and bounded below, this is equivalent to saying that $F(U_1) \rightarrow F(U_0)$ is a chain homotopy equivalence.

We have already given one definition of *polynomial of degree $\leq p$* . Namely, F is polynomial of degree $\leq p$ if the projection

$$F(U) \longrightarrow \text{holim}_{\substack{W \subset U \\ W \in \mathcal{O}_p}} F(W)$$

induces an isomorphism in homology. There is another definition in terms of cubical diagrams. Because we work in $\mathcal{O}^t(L)$ rather than $\mathcal{O}(L)$, it is more technical than [29, Def.2.2]. Given an open set $U \in \mathcal{O}(L)$ and pairwise disjoint closed

subsets C_0, \dots, C_p in $U \setminus \partial L$, we can make an S -cube $U \setminus C_\bullet$ in $\mathcal{O}(L)$, where $S = \{0, 1, \dots, p\}$, by

$$T \mapsto U \setminus \bigcup_{t \in T} C_t$$

for $T \subset S$. (An S -cube is a functor, covariant or contravariant, defined on the poset of subsets of S .) To define what it means for F on $\mathcal{O}^t(L)$ to be polynomial, we are mainly interested in the case where $U \in \mathcal{O}^t(L)$ and the C_t are pairwise disjoint tame co-handles. This means that each C_t is a smooth codimension q_t submanifold diffeomorphic to a euclidean space, and there exists a smooth compact codimension zero submanifold $K \subset U$ such that ∂K has transverse intersections with the C_t , each intersection $K \cap C_t$ is a disk of codimension q_t , and the inclusion

$$(\text{int}(K); \text{int}(K) \cap C_0, \dots, \text{int}(K) \cap C_p) \longrightarrow (U; C_0, \dots, C_p)$$

is isotopic to a diffeomorphism (by an isotopy in U which respects the indicated submanifolds). It follows that each $U \setminus C_T$ is in $\mathcal{O}^t(L)$. We say that F is polynomial of degree $\leq p$ if, in this situation, the S -cube of chain complexes

$$T \mapsto F(U \setminus C_T)$$

is always cartesian, where $C_T = \bigcup_{t \in T} C_t$. (For the concept of a cartesian S -cube, see [8].) This definition of *polynomial of degree $\leq p$* is equivalent to the earlier one by arguments given in [29], especially in the proofs of [29, Thm 4.1, Thm 5.1].

Next, let E be a *covariant* functor from $\mathcal{O}^t(L)$ to cochain complexes of abelian groups. Suppose that each $E(U)$ is homotopy equivalent to a cochain complex of finitely generated free abelian groups, bounded below. We say that E is polynomial of degree $\leq p$ if, for any U in $\mathcal{O}^t(L)$ and pairwise disjoint tame co-handles C_0, C_1, \dots, C_p in U , the S -cube of cochain complexes

$$T \mapsto F(U \setminus C_T)$$

is cocartesian, where $S = \{0, 1, \dots, p\}$ and $C_T = \bigcup_{t \in T} C_t$. (This is our main definition of *polynomial of degree $\leq p$* in the covariant setting and we studiously refrain from giving another definition which might mention $\mathcal{O}p$.)

With these definitions, the proof of our proposition turns into a triviality. Namely, F with the stated finite generation and boundedness properties is polynomial of degree $\leq p$ if and only if $E = DF$ is polynomial of degree $\leq p$, because the functor $\text{hom}(-, \mathbb{Z})$ transforms cartesian S -cubes of chain complexes into cocartesian S -cubes of cochain complexes. (Of course, we also make use of the fact that the functor $\text{hom}(-, \mathbb{Z})$ is involutory on chain complexes of finitely generated abelian groups.) \square

Corollary E.2. *Let F be a good contravariant functor from $\mathcal{O}^t(L)$ to chain complexes of abelian groups. Suppose that $F(U)$ is homotopy equivalent to a chain complex of finitely generated free abelian groups, bounded below, for every U in $\mathcal{O}^t(L)$. Then the following are equivalent:*

- (1) F is homogeneous of degree $\leq p$;
- (2) DF is homogeneous of degree $\leq p$.

Proof. The functor F of degree $\leq p$ is homogeneous of degree p if and only if $F(U)$ is contractible for every $U \in \mathcal{O}m$ where $m < p$. This can be taken as a definition. The same definition applies in the covariant case. Therefore, F is homogeneous if and only if DF is. \square

Lemma E.3. *Let F be a good contravariant functor from $\mathcal{O}^t(L)$ to based spaces, or to chain complexes of abelian groups. Suppose that $F(U)$ is m -connected for every $U \in \mathcal{O}^t(L)$. Then $T_p F(U)$ is $(m - \ell p - p)$ -connected for every $U \in \mathcal{O}^t(L)$.*

Proof. We consider first the case where the values of F are based spaces (where *space* means compactly generated space, and base points are nondegenerate). The proof is by induction on p . For the induction beginning we take $p = 0$, in which case $T_p F(U) \simeq F(Q)$ where Q is an open collar on ∂L . Since $F(Q)$ is m -connected, this proves our claim for $p = 0$. For the induction step, we assume that $p > 0$ and that $T_{p-1} F(U)$ is $(m - \ell(p-1) - p + 1)$ -connected, alias $(m - \ell p - p + \ell + 1)$ -connected. If $m - \ell p - p + \ell + 1 < 0$, then there is nothing to prove for $T_p F(U)$. If $m - \ell p - p + \ell + 1 \geq 0$, then $T_{p-1} F(U)$ is path connected. The homotopy fiber (over the base point) of

$$T_p F(U) \longrightarrow T_{p-1} F(U)$$

is a homogeneous functor of degree p in the variable U . There is a formula [29] which describes it as a space of sections, subject to boundary conditions, of a fibration with a distinguished “zero” section on

$$\binom{U \setminus \partial L}{p}.$$

The fiber over a configuration $S \subset U \setminus \partial L$ of p points is homotopy equivalent to the total homotopy fiber of the S -cube

$$R \mapsto F(N(R))$$

where R is a subset of S and $N(R)$ is a tubular neighborhood of $R \cup \partial L$ in U . Since $F(N(R))$ is always m -connected, the total homotopy fiber of those cubes is $(m - p)$ -connected. Since the configuration space has dimension ℓp , it follows that the section space is $(m - \ell p - p)$ -connected. Therefore the homotopy fiber of $T_p F(U) \rightarrow T_{p-1} F(U)$ is $(m - \ell p - p)$ -connected, and since our connectivity estimate for $T_{p-1} F(U)$ is larger, it follows that $T_p F(U)$ is also $(m - \ell p - p)$ -connected.

Now we need to look at the case where F has chain complex values. This part of the proof is more sketchy than the first. We reason that F can be viewed as a functor with spectrum values. The classification of homogeneous functors in [29] carries over to the setting of good cofunctors on $\mathcal{O}^t(L)$ or $\mathcal{O}(L)$ with values in spectra. (Fibrations over configuration spaces have to be replaced by fibered spectra over configuration spaces.) Therefore, the proof just given for the case of an F with space values carries over to the case of an F with spectrum values.

We finish by explaining how chain complexes should be viewed as spectra. In the introduction, it was suggested that the Kan-Dold equivalence can be used where necessary to view chain complexes as simplicial abelian groups, hence via geometric realization as spaces. That suggestion needs to be made more precise. The Kan-Dold construction ε takes a chain complex C_* graded over \mathbb{Z} to the simplicial abelian group $\varepsilon(C_*)$ whose set of n -simplices is the set of chain maps from the cellular chain complex of Δ^n to C_* . It is an equivalence of categories *only* when restricted to chain complexes which are zero in negative degrees. It respects homotopy limits in the sense that

$$(E.1) \quad \varepsilon\left(\operatorname{holim}_{\alpha} C_*(\alpha)\right) \simeq \operatorname{holim}_{\alpha} \varepsilon(C_* \alpha)$$

for a functor $\alpha \mapsto C_*(\alpha)$. That formula has a weakness, though, in that ε does not fully respect properties such as *not m-connected*. For example, it may happen that each $C_*(\alpha)$ is 3-connected and that $\text{holim}_\alpha C_*(\alpha)$ is (-5) -connected but not (-4) -connected. In such a case we lose essential information by applying ε . The cure is to use a variant $\underline{\varepsilon}$ of the Kan-Dold construction which takes chain complexes to CW-spectra rather than spaces. This is easy to supply. The formula of (E.1) remains valid for $\underline{\varepsilon}$. Compared with ε , the construction $\underline{\varepsilon}$ has the decisive advantage that it respects properties such as *m-connected* and *not m-connected*. \square

We continue with a technical variant of analyticity with built-in convergence estimates. This applies to functors with space values and to functors with chain complex values.

Definition E.4. [9, 4.1.10] Let F be a good cofunctor on $\mathcal{O}^t(L)$. We say that F is ρ -analytic with excess c (where $\rho, c \in \mathbb{Z}$) if it has the following property. For $U \in \mathcal{O}^t(L)$ and $j \geq 0$ and pairwise disjoint tame co-handles C_0, \dots, C_j in U , of codimension $q_t < \rho$ respectively, the cube $F(U \setminus C_\bullet)$ is $(c + \sum_t (\rho - q_t))$ -cartesian.

The corresponding definition in [9] contains a very unfortunate typo or error (where it has $k > 0$ instead of $k \geq 0$, corresponding to $j \geq 0$ above). Apart from that it is just slightly more general because tameness assumptions are absent. As in [9, 4.2.1] we have a convergence theorem.

Theorem E.5 (Convergence). *Suppose that F is ρ -analytic with excess c , and $U \in \mathcal{O}^t(L)$ has a tame handle decomposition relative to a collar on ∂L , with handles of index $\leq q$ only, where $q < \rho$. Then the canonical map*

$$F(U) \longrightarrow T_{j-1}F(U)$$

is $(c + j(\rho - q))$ -connected, for $j > 1$. Therefore the Taylor tower of F , evaluated at U , converges to $F(U)$.

(The *tame handle decomposition* for U is an ordinary handle decomposition for $K \subset U$, where $K \in \kappa(U)$ and the inclusion $\text{int}(K) \rightarrow U$ is isotopic to a diffeomorphism, relative to ∂L .)

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